Basic Calculus

Basic Calculus

From Archimedes to Newton to its Role in Science

Alexander J. Hahn
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For the Instructor

This text could as well have the title *The Story of Calculus*. It weaves a tale of mathematics from the trigonometry of the Greeks, the analytic geometry of Descartes, the calculus of Leibniz and Newton, to the role of calculus in a number of disciplines of basic science and modern economics. This calculus is elementary and current, but follows the subject's historical flow as it was driven by the important problems of the day.

The Purpose: Mathematics texts usually present their subject with a focus on a mastery of subroutines, leaving interesting connections with other disciplines to footnotes and exercises, and failing to illuminate the impact of mathematics in a persuasive manner. It is the twofold purpose of this text to develop calculus from within its rich historical context and to demonstrate its broad and formidable informative power. The subject is painted with a broader brush that includes basic elements of science, but the emphasis is always on the careful development of the mathematics and the information that it provides.

Experience at Notre Dame: My colleagues and I use this text in two different first-year calculus courses at the University of Notre Dame. One is a two-semester calculus sequence for arts and letters honors students. The other is a one-semester course of elementary applications of calculus for regular arts and letters students and architecture majors. The overall response from these students has been very positive, even though they have brought very different backgrounds and abilities to the classroom. This has been evident from the high formal course evaluations by the students as well as their supportive glowing informal comments. Indeed, I was awarded the university-wide Madden award for teaching excellence in 1995 in large measure because of their "applause." I should hasten to point out, however, that this text is not "mathematics as usual." Some students take to it easily, but others find it difficult to leave behind the "plug and chug" mentality that they bring from high school. It goes without saying that the instructor needs to be very sensitive to the nature and level of the audience. The pace of the course, the amount of reviewing, the selection of materials, the exposition of the elementary aspects of science, and the choice of homework exercises (from routine, to demanding, to horizon-expanding), are *all* factors that must be given very careful attention, because all will have considerable impact on the scope and substance of the course.

Description of the Contents: Chapter 1 is an exposition of basic trigonometry and its applications (to the concerns of the Greek thinkers). Chapter 2 has an elementary look at the mathematics and astronomy of the "Ptolemaic system." Chapter 3 introduces conic sections and area, and presents Archimedes' analysis of the area of a parabolic section. Chapter 4 develops analytic geometry and the astronomy of Galileo and Kepler. Chapters 5 and 6 together constitute a modern course in essential calculus based on the work of Leibniz and Newton. Chapter 7 focuses on the heart of Newton's *Principia Mathematica*. Chapters 8, 10, and 13 develop calculus in a more complete and definitive way, in part as a response to fundamental questions that arise in the arguments of Leibniz and Newton. Chapter 9 consists of four mathematical studies: a pulley problem of De L'Hospital, the suspension bridge, an experiment of Galileo with incline planes, and geometric optics. Chapter 11 analyzes radioactivity, three nuclear clocks and what they tell us, and the dynamics of populations. Chapter 12 presents the basics of business calculus. Chapter 14, finally, develops fundamental concepts of physics (work, energy, impulse, and momentum) and applies calculus to a variety of topics ranging from the motion of a bullet in the barrel of a gun, to the propulsion of rockets, to the expanding universe.

Logical Interdependence: This text contains several components that are essentially self-contained. One such component is the calculus in Chapters 5 and 6 together with some analytic geometry (from Chapter 4). Chapter 7 and fundmental facts about conic sections (from Chapter 3) and the calculus of velocity (from Chapter 6) is another. Chapter 9 along with a some basic calculus (the chain rule and the max/min strategy from Chapter 8) is a third component. A fourth is Chapter 11 and some elementary properties of exponential and logarithm functions (from Chapter 10). A fifth component consists of the business calculus of Chapter 12 together with very basic differential and integral calculus (as covered in Chapters 5 and 8). Finally, Chapters 13 and 14 together comprise a challenging component of integral calculus and physics.

Examples of Courses from This Text: A number of different first-year calculus courses can be taught from this text to audiences ranging from arts and letters to science students. Some specific suggestions follow:

Course A: This fast-paced, one-semester course proceeds linearly from Chapter 1 through Chapter 7. It covers basic college mathematics from geometry to trigonometry and from analytic geometry to calculus. It does so from "Archimedes to Newton," in other words, from within the context of the scientific questions that helped drive these disciplines. This outline has been followed in the first semester of the arts and letters honors course at Notre Dame. A colleague has taught this course with Chapter 8 in place of Chapter 7. The contrast between the calculus of Leibniz/Newton and the more definitive approach of Chapter 8 makes it possible to discuss the rationale for rigor in mathematics.

A less demanding version of the outline proceeds as follows:

Course B: The approaches of Leibniz and Newton in Chapters 5 and 6 together provide a modern course of differential and integral calculus. This course is intuitive, devoid of excess baggage, concentrates on the essence of the matter, and includes important applications. If analytic geometry needs attention, add Section 4.3. This core material can be supplemented in a number of ways. The origins of integral calculus can be traced by studying Archimedes' analysis in Chapter 3. After a review of the conic sections, also in Chapter 3, the students are ready for the challenging look at the heart of Newton's *Principia* that Chapter 7 undertakes.

A more traditional two-semester calculus course for science students can be taught using the following model:

Course C: Start with a review of the conic sections from Chapter 3 and continue with the analytic geometry (and its applications to astronomy) of Chapter 4. Then cover Chapters 8 to 11 and Chapters 13 and 14. These chapters provide a traditional calculus course with a thorough look at applications. A number of different topics in science are linked together by the common thread of calculus that gives essential information to them all. The many exercises can be used to expand the scope of topics.

A toned-down version of this outline is the basis of a second-semester course of elementary applications of calculus taught to arts and letters students and architecture majors at Notre Dame:

Course D: This course has as its single goal the illustration of the proposition that calculus informs a variety of different disciplines with absolutely crucial information. The course has any basic calculus course as prerequisite and (with insertions of appropriate reviews) focuses on a selection of applications that can include the pulley problem of L'Hospital, radioactivity and nuclear clocks, models of populations, bullets in rifles, and Hubble's law and Einstein's universe.

Course E: An essential one-semester "business calculus" can be taught from Chapters 5, 6, 8, and 12.

Because there is such a rich historical discussion throughout the text, judicious selections of chapters can provide the basis for a variety of specialized courses for specific programs of study. For example, astronomy is a recurring theme in this book, and the materials in Chapters 1, 2, 4, 7, and 14 can serve as the basis of a course "Elements of mathematics in astronomy." Chapters 4, 5, 6, and 7 along with appropriate reading materials, such as those authored by Herbert Butterfield and Thomas Kuhn, form the substance of a course "revolutions in science and mathematics." A slower and more elementary "mathematics and science of the Greeks" could flow from Chapters 1, 2, and 3 and Section 4.1 and selected reading from Aristotle and Plato.

All of the courses just discussed can be supplemented with the web site

`http://www.nd.edu:80/-hahn/`

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Alexander J. Hahn
Notre Dame, 1998

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Part I

From Archimedes to Newton

1

The Greeks Measure the Universe

This chapter is an exposition of fundamental matters that will be in use throughout this book. It introduces basic concepts of geometry and trigonometry and shows how the Greeks—who developed these disciplines into usable theories—put them to use.

1.1 The Pythagoreans Measure Length

The primary purpose of a number system is to count and measure things. This is where we will begin. The Pythagorean theorem[1] asserts that in a right triangle (see Figure 1.1), the equality $a^2 + b^2 = c^2$ relates the lengths of the sides. We are assuming that some unit of length (nowadays it would be a foot, yard, mile, inch, or meter, for example) is given. The Pythagorean theorem works "in reverse" also. Namely, if in some triangle the lengths of the sides satisfy the relationship $a^2 + b^2 = c^2$, then the triangle is a right triangle with c the length of the hypotenuse and a and b the lengths of the other two sides.

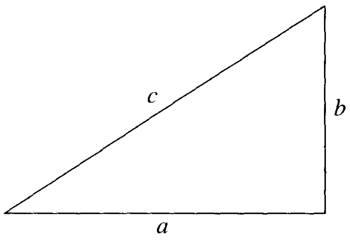


Figure 1.1

It appears that "rope stretchers" used this fact in early construction. Take, for example, a rope that has a length of 12 units (Figure 1.2): If the rope is stretched out horizontally on the ground in the triangular pattern indicated in Figure 1.3, then, since the lengths of the sides satisfy the relationship $4^2 + 3^2 = 16 + 9 = 25 = 5^2$, the Pythagorean theorem asserts that the angle at 4 is a right angle. The construction of two perpendicular walls can begin!

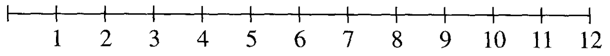


Figure 1.2

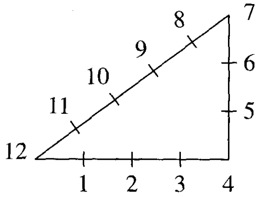


Figure 1.3

Assume that μ is some unit of length. We will say that a straight segment is *measurable* in μ if its length in units of μ is $\frac{m}{n}$, where m and n are both positive integers. So the segment is measurable if its length can be expressed precisely in whole or fractional units of μ. For instance, if the unit μ is the inch, then the segments of lengths $26\frac{4}{7} = \frac{186}{7}$ inches, as well as $53\frac{14}{29} = \frac{1551}{29}$ inches and $1726\frac{951}{3657} = \frac{6312933}{3657}$ inches are measurable.

The question presents itself as to whether all straight segments are measurable in the unit μ. Assume that the segment pictured directly below is measurable and that it has length $\frac{m}{n}$.

Take another segment of the same length, place them together at right angles, and form the right triangle pictured in Figure 1.4. Is the resulting hypotenuse h measurable? We will see (surprise?) that it is not.

Assume that it is. Then the length of h is some fraction $\frac{r}{s}$ of positive integers. By Pythagoras's theorem, $\left(\frac{r}{s}\right)^2 = \left(\frac{m}{n}\right)^2 + \left(\frac{m}{n}\right)^2 = 2\left(\frac{m}{n}\right)^2$. By clearing denominators, $r^2n^2 = 2m^2s^2$. Now set $x = rn$ and $y = ms$, and

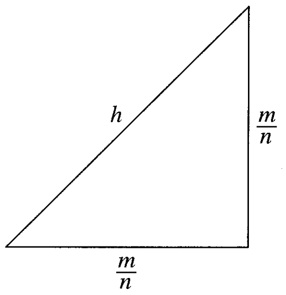


Figure 1.4

observe that $x^2 = 2y^2$. Factoring 2 out of x as many times as possible gives $x = 2^a x_1$ with x_1 odd. In the same way, $y = 2^b y_1$ with y_1 odd. (If, for instance, x were equal to 172, then $172 = 2 \cdot 86 = 2 \cdot 2 \cdot 43 = 2^2 \cdot 43$. So $a = 2$ and $x_1 = 43$.) By a substitution we get $(2^a x_1)^2 = 2(2^b y_1)^2$. So by the rules for exponentiating,

$$2^{2a} x_1^2 = 2^{2b+1} y_1^2.$$

Notice that $2a \neq 2b + 1$, since $2a$ is even and $2b + 1$ is odd. Therefore, either $2a > 2b + 1$ or $2a < 2b + 1$.

1. Assume that $2a > 2b + 1$. So $2^{2a-(2b+1)} x_1^2 = y_1^2$. It follows that y_1^2 is even. Since y_1 is odd, it has the form $y_1 = 2k + 1$ for some integer k. So $y_1^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$, which is the sum of an even integer and 1. But this means that y_1^2 is odd. Thus y_1^2 is both even and odd. This is impossible.
2. Assume next that $2a < 2b + 1$. This implies that $x_1^2 = 2^{(2b+1)-2a} y_1^2$. By the argument in Step 1, it is now x_1^2 that is both even and odd. This too is impossible.

Reflect over what has been done. The discussion started with the supposition that h is measurable in μ. Then the argument moved in a strictly logical way to impossible consequences. The inescapable conclusion is that h cannot be measurable in μ.

The preceding proof was not taken from an old Greek text. It is, however, from the point of view of the precision of the logic and the flow of its argument, very much in the spirit of Greek mathematics. The point is this: While the hypotenuse h is a perfectly valid geometric construction, it cannot be measured with the numbers of the Pythagoreans. In particular, their numerical considerations ran into limitations that geometrical ones did not. The suggestion presents itself that this was an important reason why Greek geometry and trigonometry flourished in a way that numerical analyses and algebra did not.

Pythagoras and his followers had founded a Greek colony in today's southern Italy in the 6th century B.C. They formed a cult based on the philosophical principle that mathematics is the underlying explanation of all things from the relationship between musical notes to the movements of the planets of the solar system. Indeed, they held that all reality finds its ultimate explanation in numbers and mathematics. As we have just seen, however, the numbers of the Pythagoreans were unable to come to grips with the very basic matter of measuring length. It is believed that when the Pythagoreans realized this, a crisis ensued within their ranks that contributed to their downfall.[2] In any case, it seems somewhat ironic that the Pythagorean theorem—an assertion about the lengths of certain segments—derives its name from a school (or person) that did not possess a number system with which length could always be measured.

Today we can put it this way: The Pythagorean number system did not have enough numbers. It consisted only of the numbers of the form $\frac{m}{n}$, i.e., the rational numbers, and it did not include other real numbers. In fact, the preceding demonstration of the nonmeasurability of h shows that $\sqrt{2}$ is a number that is not rational. (Take $m = n = 1$.) It is, in other words, irrational. On the other hand, we know that

$$\begin{aligned}\sqrt{2} &= 1.4142\ldots\\ &= 1 + 4\frac{1}{10} + 1\frac{1}{100} + 4\frac{1}{1000} + 2\frac{1}{10000} + \ldots,\end{aligned}$$

and hence that $\sqrt{2}$ can be constructed in terms of a decimal expansion. This infinite process gives rise to the number line. Fix a unit of length and take a straight line that runs infinitely in both directions.

Fix a point and label it 0. Mark off a point one unit to the right of 0 and label it 1. Continue in this way to get 2, 3, ... Do a similar thing on the left of 0, but use $-1, -2, \ldots$ to label the points. Continue in this way with tenths of units, hundredths of units, and so on, to achieve the following relationship: Every point on the line corresponds to a real number, in other words, a number given by a decimal expansion, and every such number corresponds to a point on the line. See the illustration for $\sqrt{2} = 1.4142\ldots$ in Figure 1.5.

The Greeks shied away from infinite processes and did not hit upon the construction of the real numbers. (This did not occur until the 16th century!) This means that they did not have the coordinate—also called Cartesian—plane. Therefore, they could not graph algebraic equations and make use of the interplay between algebra and geometry that lies at the core of modern mathematical analysis. In this course we will not shy away from real numbers: we will make use of them throughout.

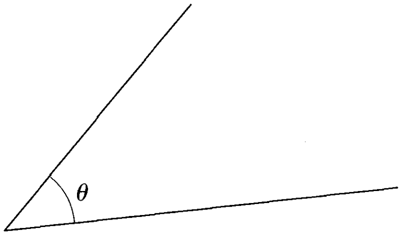


Figure 1.6

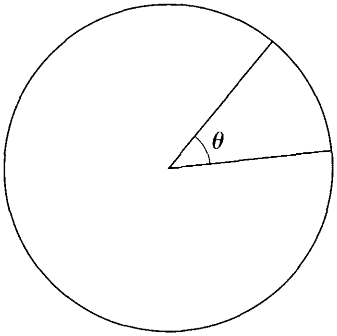


Figure 1.7

1.2 The Measure of Angles

We turn next to the study of angles. Take two line segments with a common endpoint; they form a wedge. The issue is to devise a numerical way of measuring the angle, i.e., the amount θ of the opening of the wedge. See Figure 1.6.

Place the wedge in a circle with the common point at the center. See Figure 1.7. One way of measuring θ is with degrees. This unit is designated by ° and is a legacy of the base 60 number system of the Babylonians. Declare the entire circle to have 360 degrees and then apportion degrees proportionally. So a wedge consisting of half the circle will have 180°,

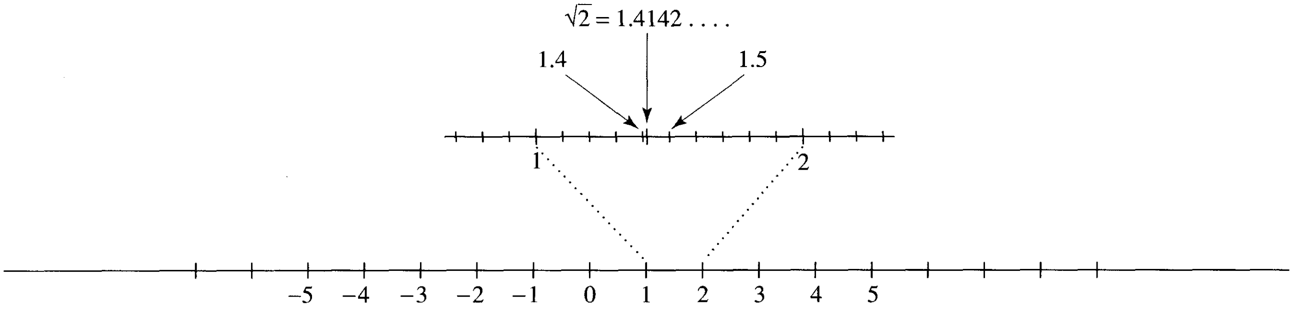


Figure 1.5

one consisting of a quarter circle will measure 90°, and so on. In this way, any angle can be assigned a certain number of degrees. It is known that the angles of any triangle add up to 180°.

A more useful numerical measure of an angle is the radian measure. It is based on the measurement of length. Let r be the radius of the circle in Figure 1.8 and let s be the length of the arc that is cut out of the circumference by the wedge. The radian measure of the angle θ is defined to be the ratio $\frac{s}{r}$. We will write $\theta = \frac{s}{r}$. An immediate question that arises is this: Does the radian measure of the angle depend on the size of the circle into which it is placed? If this measure is to be a meaningful concept, it should not.

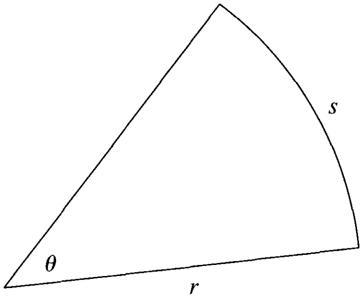


Figure 1.8

Place θ into another circle. Let R be its radius and S the length of the arc that θ cuts out. See Figure 1.9. Is $\frac{s}{r} = \frac{S}{R}$? This is the question that must be addressed. Consider what follows to be a "thought experiment," rather than something that is carried out in practice.

Let n be a positive integer (n can be, say, 5 or 500 or 40,000), and partition the wedge into n equal pieces. Connect the intersection points with straight line segments and let their respective lengths be d_n and D_n. (Note that these lengths depend on the n that you have picked.) The case $n = 4$ is shown in Figure 1.10. Observe that each of the smaller triangular wedges (those with sides of length r) is similar to each of the larger ones (with sides of length R). This is because the corresponding angles are equal. It follows that $\frac{d_4}{r} = \frac{D_4}{R}$. In the same way, $\frac{d_n}{r} = \frac{D_n}{R}$ for any n. Observe next that if n is taken to be large, then the lengths of the n segments each of length d_n add up to approximately s. So the number nd_n is nearly equal to s, and therefore $\frac{nd_n}{r}$ is nearly equal to $\frac{s}{r}$. If n is taken sequentially larger and larger, the numbers $\frac{nd_n}{r}$ close in on $\frac{s}{r}$. We abbreviate this by writing

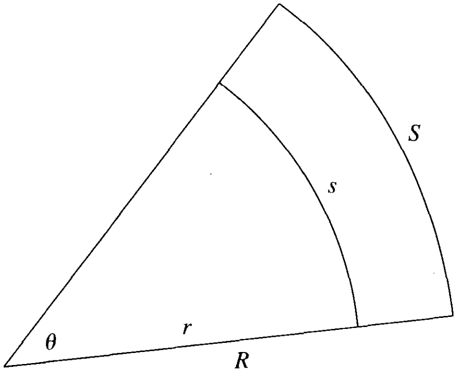


Figure 1.9

$$\lim_{n\to\infty} \frac{nd_n}{r} = \frac{s}{r}.$$

The symbol lim is short for limit and refers to the closing-in process. The symbol ∞ represents "infinity"

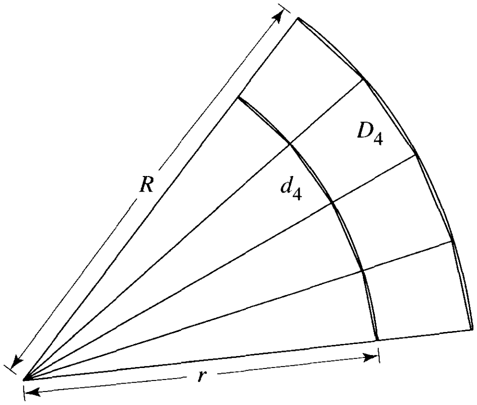


Figure 1.10

and $\lim_{n\to\infty}$ means that n, by being taken larger, is being "pushed to infinity." Proceeding in exactly the same way with the larger arc, we get that

$$\lim_{n\to\infty} \frac{nD_n}{R} = \frac{S}{R}.$$

Since $\frac{d_n}{r} = \frac{D_n}{R}$ for any n, we see that

$$\frac{s}{r} = \lim_{n\to\infty} \frac{nd_n}{r} = \lim_{n\to\infty} \frac{nD_n}{R} = \frac{S}{R}.$$

So our thought experiment has established the required equality $\frac{s}{r} = \frac{S}{R}$. Observe that the radian measure of an angle is a ratio of lengths. It is therefore a real number.

We will learn later that "limit" procedures such as the one we just considered are the cornerstone of calculus.

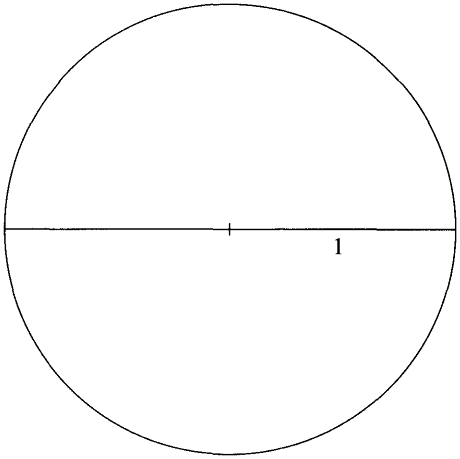


Figure 1.11

Now take a circle of radius 1 (Figure 1.11). The length of one-half of its circumference is a number that is denoted by π. Consider the angle 180° and observe that its radian measure is $\frac{\pi}{1} = \pi$. Now take a circle of radius r and let c be its circumference (Figure 1.12). Using this circle, we get that the radian measure of 180° is equal to $\dfrac{c/2}{r}$. Since the radian measure of an angle is the same regardless of which circle is used, it follows that $\dfrac{c/2}{r} = \pi$. This gives the well-known formula

$$c = 2\pi r$$

for the circumference of a circle of radius r.

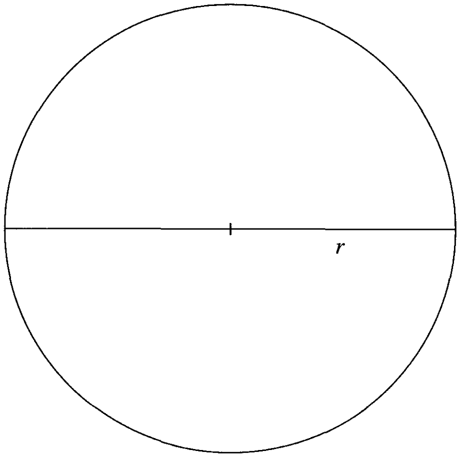


Figure 1.12

Consider the circle of Figure 1.11 and check that degrees and radians correspond as shown in Table 1.1.

Table 1.1

Degrees	Radians
360°	2π
180°	π
90°	$\frac{\pi}{2}$
60°	$\frac{\pi}{3}$
45°	$\frac{\pi}{4}$
30°	$\frac{\pi}{6}$
1°	$\frac{\pi}{180}$

Since π represents length, it is some real number. What is its decimal expansion?[3] Take a circle of radius 1 and center C and divide the upper semicircle into three equal parts, as shown in Figure 1.13. Each of the angles is equal to $\frac{\pi}{3}$. Concentrate on the wedge on the right. Let A and B be the indicated points of intersection, and observe that the isosceles triangle ΔCAB is equilateral. (Why?) Therefore, AB has length 1. Choose the point P on the circle such that the segment CP bisects the angle $\angle ACB$ and draw the tangent $A'B'$ to the circle at P. Since the bisector CP and the tangent $A'B'$ are perpendicular (by a basic property of the circle), the angles at A' and B' are both equal to $\frac{\pi}{3}$. It follows that the triangles $\Delta CA'P$ and $\Delta CB'P$ are equal. Put[4] $A'P = PB' = t$. Since the three angles of $\Delta CA'B'$ are equal, it follows that $\Delta CA'B'$ is equilateral. Therefore, CA' has length $2t$, and by the Pythagorean theorem applied to $\Delta CPA'$, it follows that $1 + t^2 = (2t)^2$. Thus $3t^2 = 1$, and hence $t = \frac{1}{\sqrt{3}}$. So $A'B'$ has length $2t = \frac{2}{\sqrt{3}}$. Since the radian measure of the angle $\frac{\pi}{3}$ is the length of the arc AB, it seems plausible (we will be content with plausibility in this discussion—see also Exercise 15) from Figure 1.13, that

$$1 = AB < \frac{\pi}{3} = \text{length arc } AB < A'B' = \frac{2}{\sqrt{3}}.$$

Now multiply through by 3 to get

$$3 < \pi < \frac{2 \cdot 3}{\sqrt{3}} = \frac{2\sqrt{3}\sqrt{3}}{\sqrt{3}} = 2\sqrt{3}.$$

This corresponds to $3 < \pi < 3.47$.

Archimedes (287–212 B.C.)—we will encounter him again soon—used an argument, similar in principle but much more involved (it uses many more triangles instead of our three), to show that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}.$$

This gives $3.1408 < \pi < 3.1429$. The correct expansion begins $\pi = 3.14159\ldots$. It turns out (and this is difficult to establish) that π is an irrational number.

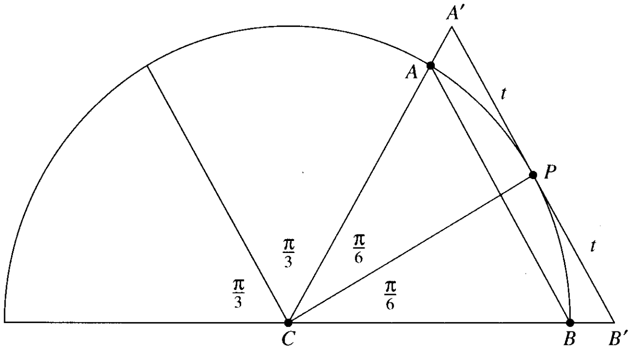


Figure 1.13

1.3 Eratosthenes Measures the Earth

Eratosthenes (276–194 B.C.) was director of the Museum of Alexandria, as Euclid had been before him. Under the sponsorship of Egypt's Ptolemaic rulers, this was an early version of a government funded research institute.

It was the commonly accepted view among Greek philosophers that the Earth is round. Eratosthenes was able to measure its size. He knew that at noon

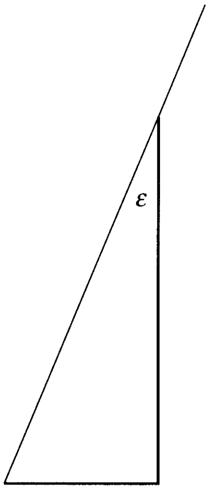


Figure 1.14

at the summer solstice of every year (this occurs on or about June 21 in the Northern Hemisphere) in the town of Syene (near today's Aswan in Egypt), the Sun shines into the very bottom of a deep well. This means that at precisely that time of the year the Sun is directly overhead at this location. In Alexandria, again precisely at noon at the summer solstice, Eratosthenes measured the length of the shadow of a gnomon[5] in vertical position (using a plumbline) and determined the angle ε in Figure 1.14 to be $7.5°$. Eratosthenes knew also that Syene was (roughly) due south of Alexandria at a distance of about 500 miles.[6]

He next let r be the radius of the Earth and noticed that the basic situation is given by Figure 1.15 with $\varepsilon = 7.5°$ and $s = 500$. The rest was easy. On the one hand,

$$\varepsilon = \left(7\frac{1}{2}\right)\frac{\pi}{180} = \frac{15(3.14)}{360} = 0.13 \text{ radians.}$$

But on the other hand, $\varepsilon = \frac{s}{r} = \frac{500}{r}$. Therefore,

$$r = \frac{500}{\varepsilon} = \frac{500}{0.13} = 3850 \text{ miles.}$$

So Eratosthenes—quite literally with a stick, some observations, and geometry, cemented together by pure thought—had measured the size of the Earth! Of course, he had only an approximation. The correct value of the radius of the Earth is 3950 miles.

1.4 Right Triangles

Trigonometry is the study of the right triangle and the applications of this study. Indeed, the word trigonometry is the Greek rendition of "measuring the triangle." This section recalls some ba-

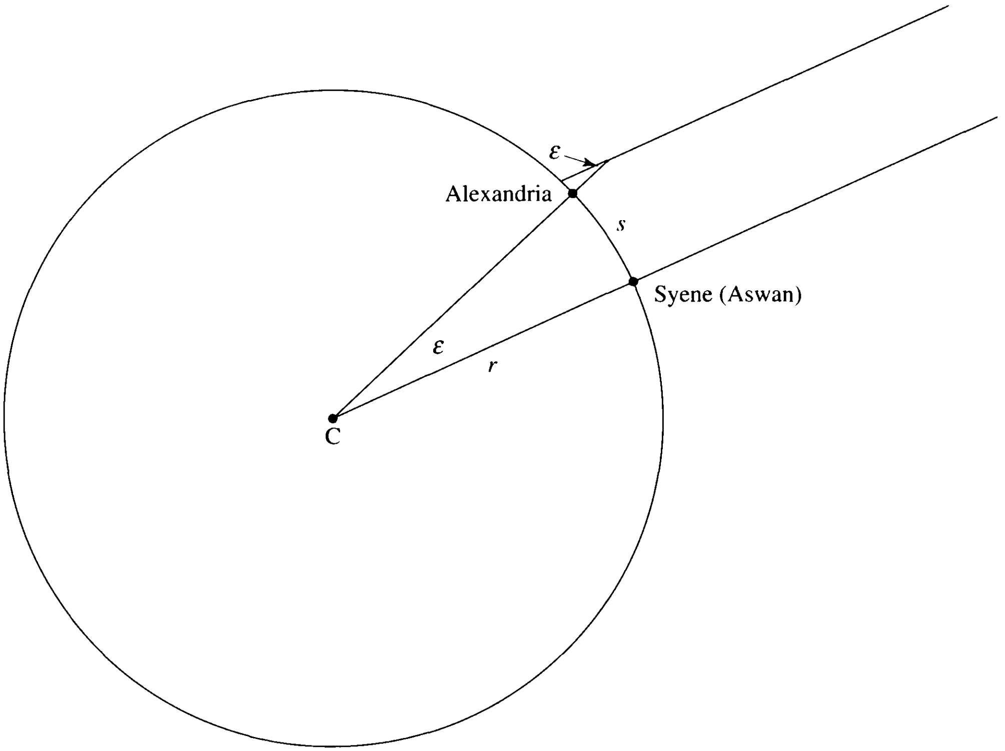


Figure 1.15

sic facts, all known—with different terminology and notation—to the Greeks.

Consider the right triangle in Figure 1.16. For the given angle θ, define the "sine," "cosine," and "tangent" to be the following ratios of lengths:

$$\sin\theta = \frac{a}{h}, \quad \cos\theta = \frac{b}{h},$$

and

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{a}{h}}{\frac{b}{h}} = \frac{a}{b}.$$

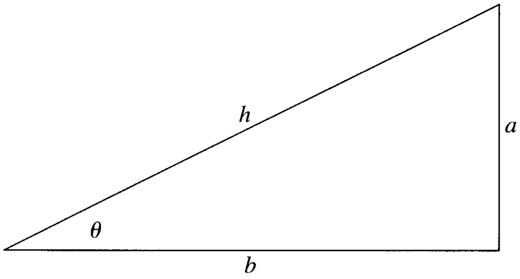


Figure 1.16

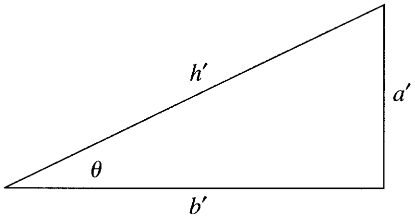


Figure 1.17

Only the size of the angle θ matters in these definitions and not the right triangle into which it is placed. This is easily seen as follows. Suppose another right triangle is used; see Figure 1.17. Superimposing the two triangles shows that they are similar. (See Figure 1.18.) Therefore, $\frac{a'}{h'} = \frac{a}{h}$, $\frac{b'}{h'} = \frac{b}{h}$, and $\frac{a'}{b'} = \frac{a}{b}$, so it doesn't matter which right triangle is used to compute $\sin\theta$, $\cos\theta$, and $\tan\theta$.

There are many identities that relate $\sin\theta$, $\cos\theta$, and $\tan\theta$. For example, by Pythagoras's theorem, $a^2 + b^2 = h^2$. So $\frac{a^2}{h^2} + \frac{b^2}{h^2} = 1$, and therefore,

$$\sin^2\theta + \cos^2\theta = 1.$$

It is customary to write $\sin^2\theta$ instead of the less efficient $(\sin\theta)^2$.

We will now compute $\sin\theta$, $\cos\theta$, and $\tan\theta$ for some standard values of θ. Consider the right triangle in Figure 1.19. Since it is isosceles, the acute angles are each 45° or $\frac{\pi}{4}$ radians. Therefore,

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \ \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \text{ and } \tan\frac{\pi}{4} = \frac{1}{1} = 1.$$

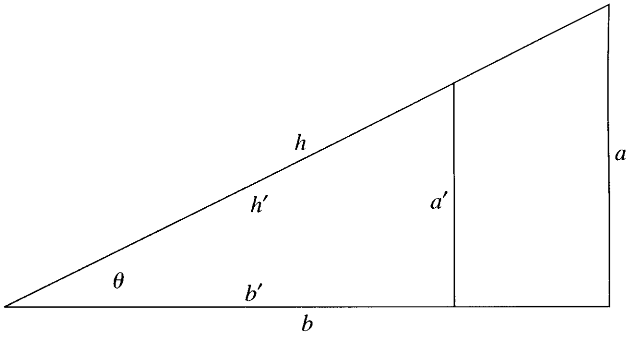


Figure 1.18

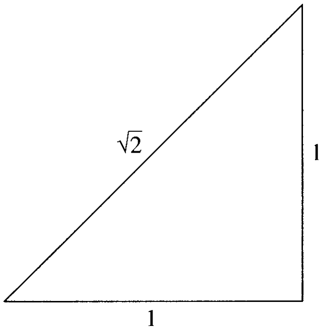


Figure 1.19

Next, take the equilateral triangle that has sides of length 2 and height h. By Pythagoras's theorem, $h^2 + 1^2 = 2^2$. So $h^2 = 3$ and $h = \sqrt{3}$. See Figure 1.20. Recalling that the angles of an equilateral triangle are each equal to 60° or $\frac{\pi}{3}$ radians, we see that

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},\ \cos\frac{\pi}{3} = \frac{1}{2},\ \text{and}\ \tan\frac{\pi}{3} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

Since each of the smaller angles at the top is $\frac{\pi}{6}$, it follows that

$$\sin\frac{\pi}{6} = \frac{1}{2},\ \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2},\ \text{and}\ \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

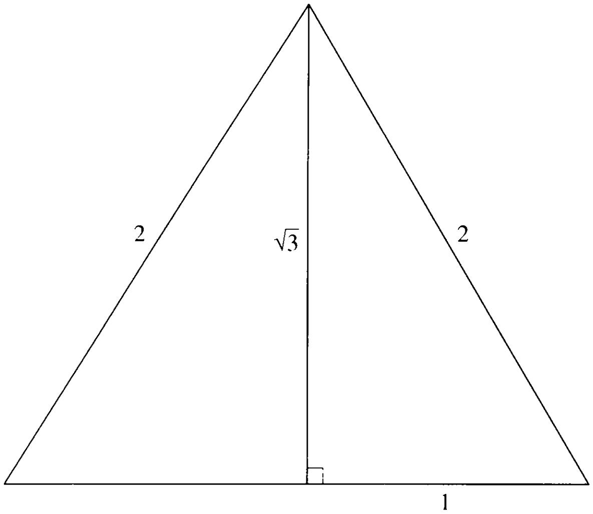


Figure 1.20

Observe that $\sin 0$, $\cos 0$, and $\tan 0$ do not make sense, for the simple reason that there is no right triangle with an angle of 0° or, equivalently, 0 radians. Suppose, however, that θ is very small. A look at the right triangle with hypotenuse 1 in Figure 1.21 shows that $\sin\theta = \frac{a}{1} = a$ is also very small. So if θ is close to 0, then $\sin\theta$ is also close to zero. Take θ sequentially smaller and smaller. This rotates the hypotenuse downwards and pushes a to zero. Therefore, $\sin\theta = a$ is pushed to zero. We summarize this by writing

$$\lim_{\theta\to 0}\sin\theta = 0.$$

It now makes sense to set $\sin 0 = 0$. For entirely similar reasons, $\sin\frac{\pi}{2}$ does not make sense either. This time focus on the angle ϕ in Figure 1.21. As the hypotenuse rotates, ϕ is pushed to $\frac{\pi}{2}$, and in the process, $\sin\phi = \frac{b}{1} = b$ is pushed to 1. In limit notation,

$$\lim_{\phi\to\frac{\pi}{2}}\sin\phi = 1.$$

So it makes sense to set $\sin\frac{\pi}{2} = 1$. A similar analysis for the cosine shows that $\lim\limits_{\theta\to 0}\cos\theta = 1$ and $\lim\limits_{\theta\to\frac{\pi}{2}}\cos\theta = 0$; so we set $\cos 0 = 1$ and $\cos\frac{\pi}{2} = 0$.

Table 1.2

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	?

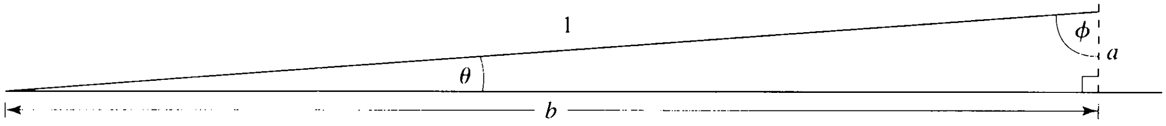


Figure 1.21

Table 1.2 summarizes this information. The reader is asked to fill in the question mark.

We conclude with some basic properties of the sine. Consider the circle of radius 1 and the right triangle shown in Figure 1.22. Observe that $\sin\theta = \frac{a}{1} = a$ and $\theta = \frac{s}{1} = s$ radians. Since $s > a$, it follows that

$$\theta > \sin\theta.$$

If θ is small, note that the lengths s and a will be close to each other. We symbolize this by writing $s \approx a$. Therefore, for small θ,

$$\sin\theta \approx \theta.$$

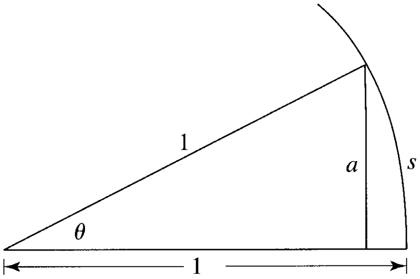


Figure 1.22

Table 1.3

θ **in degrees**	θ **in radians**	$\sin\theta$
30°	$\frac{\pi}{6} = 0.5237$	0.5000
15°	0.2618	0.2588
10°	0.1746	0.1737
3°	0.0524	0.0524
2°	0.0349	0.0349
1°	0.0175	0.0175

Take $\pi = 3.142$ and use your calculator to verify the data in Table 1.3. Observe that there is close agreement between $\sin\theta$ and θ, but only if θ is taken in radians. So when computing $\sin\theta$, make sure that your calculator is in "radian mode" and not "degree mode."

1.5 Aristarchus Sizes Up the Universe

Not much is known about Aristarchus of Samos (310–230 B.C.). He received his education directly or indirectly from Aristotle's Lyceum (the institute in Athens). The most important fact about him is this: He believed that the universe is Sun centered (or heliocentric), that the Sun is fixed, and that the Earth revolves around the Sun and rotates about its own axis in the process.

What will interest us is Aristarchus's use of "cosmic" trigonometry in his treatise *On the Magnitudes and Distances of the Sun and Moon.* His analysis rests on the following hypotheses and observations[7]:

A. The Moon receives its light from the Sun.

B. The Moon revolves in a circle about the Earth with the Earth at the center.

C. When an observer on Earth looks out at a precise half moon, the angle $\angle EMS$ in Figure 1.23 is 90°. At that moment the angle $\angle MES$ can be measured to equal 87°.

D. At the instant of a total eclipse of the Sun, the Moon and the Sun (as viewed from the Earth) subtend the same angle, and this angle can be measured to be 2°. Refer to Figure 1.24.

E. During a lunar eclipse, the shadow indicated in Figure 1.25 has width 4 times the radius r_M of the Moon. (This was based on how long the Moon was observed to be in the Earth's shadow.)

What did Aristarchus deduce from these observations? Let

r_E = the radius of the Earth

r_M = the radius of the Moon

r_S = the radius of the Sun

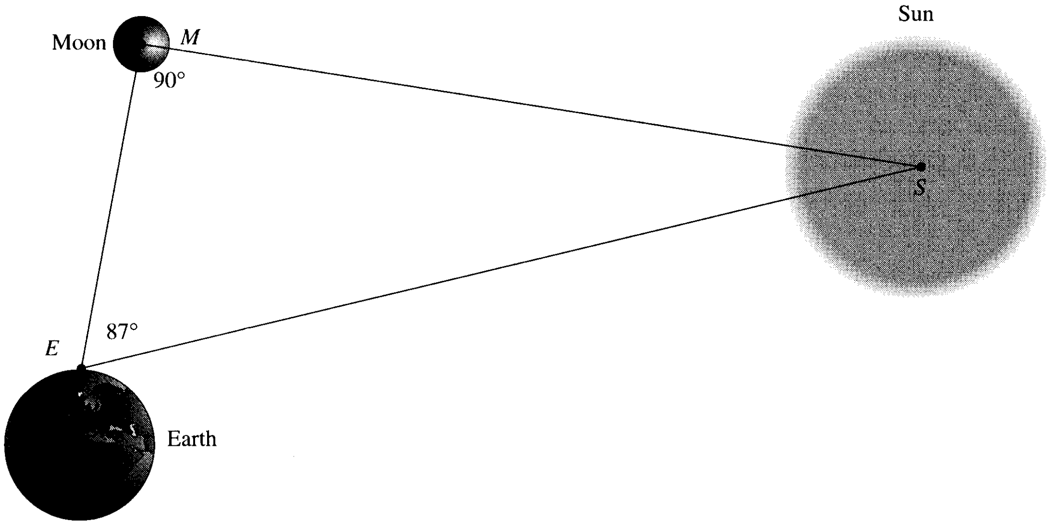


Figure 1.23

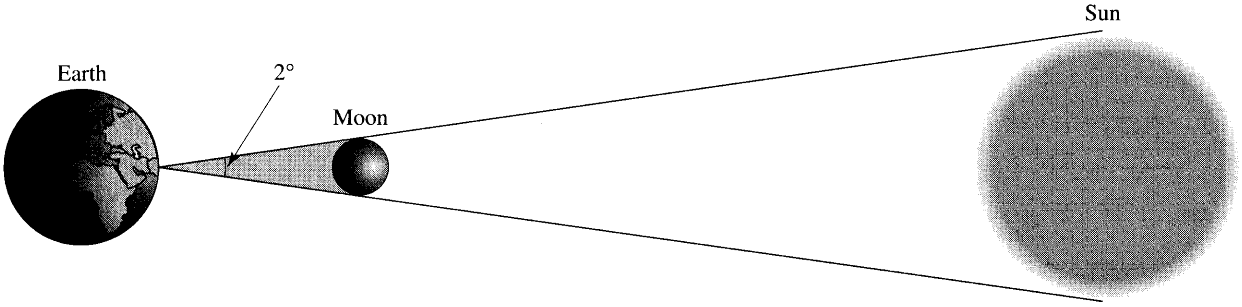


Figure 1.24

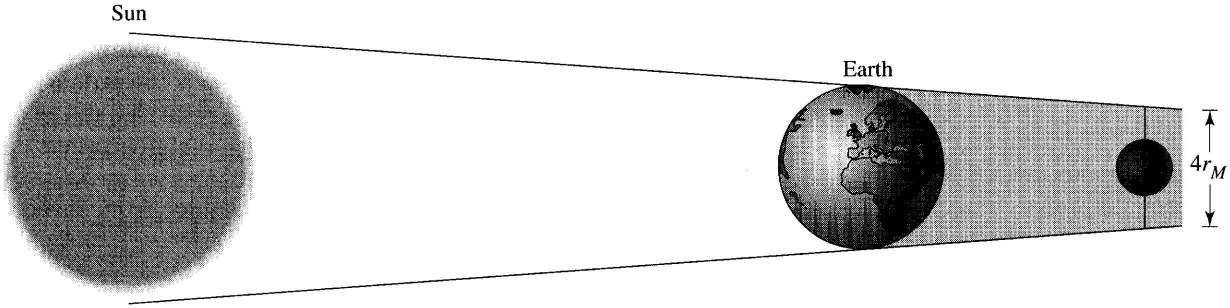


Figure 1.25

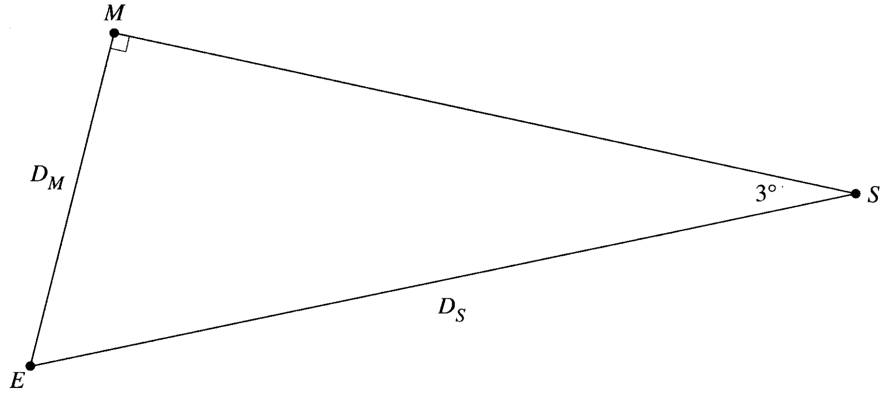


Figure 1.26

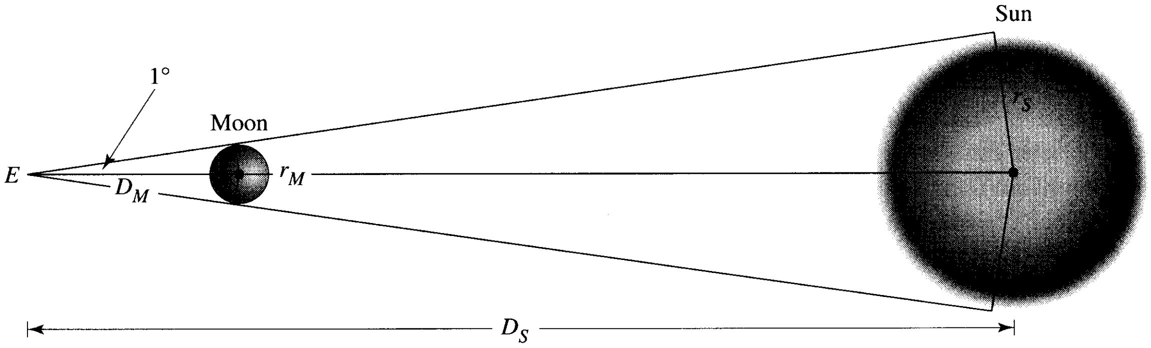


Figure 1.27

D_M = the distance from the Earth to the Moon

D_S = the distance from the Earth to the Sun

Figure 1.26 comes directly from observation **C**. Observe that $3° = \frac{\pi}{60}$. Therefore, $\frac{D_M}{D_S} = \sin 3° = \sin\frac{\pi}{60}$. Since $3° = \frac{\pi}{60}$ radians is a small angle, $\sin\frac{\pi}{60}$ is approximately equal to $\frac{\pi}{60}$. (See Table 1.3.) In view of Archimedes's estimate, we take $\pi = 3.14$. (Aristarchus was a contemporary of Archimedes and would certainly have been aware of such estimates.) This gives $\frac{\pi}{60} = 0.052$. To make things simple, we will take $\sin\frac{\pi}{60} = \frac{\pi}{60} = 0.05 = \frac{1}{20}$. So $\frac{D_M}{D_S} = \sin\frac{\pi}{60} = \frac{1}{20}$. In this way Aristarchus arrived at the estimate

$$\frac{D_S}{D_M} = 20.$$

Refer next to observation **D** and Figure 1.24. This is the situation of the solar eclipse. Figure 1.27 is an elaboration of Figure 1.24. Radii of the Moon and the Sun have been inserted and some distances have been labeled. The angle at E indicated as being equal to 1° is obtained by bisecting the 2° angle of Figure 1.24. Notice that the Sun is larger than the Moon. By similar triangles, $\frac{r_S}{r_M} = \frac{D_S}{D_M}$, and therefore

$$\frac{r_S}{r_M} = 20.$$

Observe also that $\frac{r_M}{D_M} = \sin 1°$. Since $1° = \frac{\pi}{180}$ radians, $\frac{r_M}{D_M} = \sin\frac{\pi}{180} \approx \frac{\pi}{180}$. Taking $\pi = 3.14$, it follows that $\frac{\pi}{180} = 0.017$. Note that $\frac{1}{60}$ is very close to 0.017. So to make things simple, we will take $\frac{r_M}{D_M} = \sin\frac{\pi}{180} = \frac{1}{60}$.

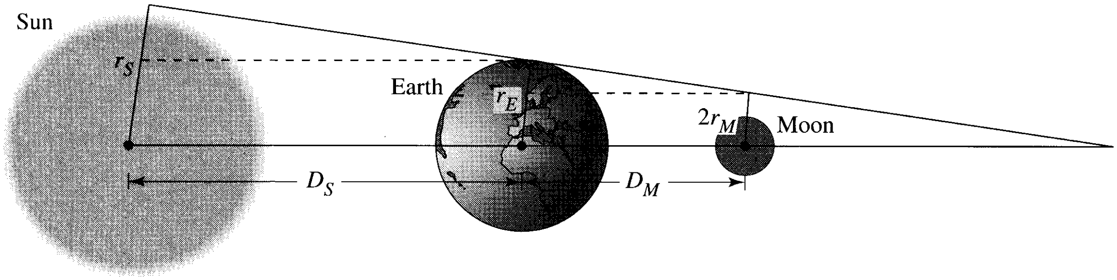


Figure 1.28

So Aristarchus obtained the approximation

$$\frac{D_M}{r_M} = 60.$$

From observation **E** and Figure 1.25, he obtained Figure 1.28. This figure shows a light ray that is tangent to both the Sun and the Earth. The radii of the Sun and the Earth drawn into the figure are both perpendicular to this light ray. The extension of the radius of the Moon indicated in the figure is perpendicular to this light ray as well. Because the two triangles with the "dotted" bases are similar, it follows that

$$\frac{r_E - 2r_M}{r_S - r_E} = \frac{D_M}{D_S}.$$

Recalling that

$$\frac{D_M}{D_S} = \frac{r_M}{r_S}$$

we get

$$\frac{r_E - 2r_M}{r_S - r_E} = \frac{r_M}{r_S}.$$

After cross-multiplying, $r_S r_E - 2r_S r_M = r_M r_S - r_M r_E$. So $r_S r_E + r_M r_E = 3r_S r_M$. Dividing this last equation through by $r_S r_M$ gives us $\frac{r_E}{r_M} + \frac{r_E}{r_S} = 3$. Since $\frac{r_S}{r_M} = 20$, we have $r_S = 20r_M$. By substitution, we get

$$3 = \frac{r_E}{r_M} + \frac{r_E}{20r_M} = \frac{20r_E + r_E}{20r_M} = \frac{21r_E}{20r_M}.$$

Therefore, $\frac{r_E}{r_M} = \frac{60}{21} = \frac{20}{7}$, and

$$r_M = \frac{7}{20} r_E.$$

Since $\frac{r_S}{r_E} = \frac{r_S}{r_M} \cdot \frac{r_M}{r_E} = 20 \cdot \frac{7}{20} = 7$, it follows that

$$r_S = 7r_E.$$

With Eratosthenes's value of $r_E = 3850$ miles, Aristarchus got the approximations

$$r_M = 1350 \text{ miles} \quad \text{and} \quad r_S = 27{,}000 \text{ miles}.$$

Since $\frac{D_M}{r_M} = 60$, he estimated that $D_M = 60r_M = 80{,}000$ miles; and inserting this value into $\frac{D_S}{D_M} = 20$, he got

$$D_S = 1{,}600{,}000 \text{ miles}.$$

Aristarchus's argument was actually more elaborate and complicated than this. He used different approximations and got slightly different answers. For example, instead of $\frac{r_M}{r_S} = \frac{1}{20}$, he obtained $\frac{1}{20} < \frac{r_M}{r_S} < \frac{1}{18}$; and instead of $\frac{r_M}{D_M} = \frac{1}{60}$, he had $\frac{1}{60} < \frac{r_M}{D_M} < \frac{1}{45}$. However, the essence of his analysis has been retained.

Table 1.4 compares Aristarchus's estimates to modern values. Notice that Aristarchus's value for the radius of the Moon is rather accurate. However, his estimates for the distance to the Moon, the radius of the Sun, and its distance from the Earth are off by factors of 3, 16, and 50, respectively.

Table 1.4

	Aristarchus	Actual
r_E radius of Earth	3850 miles[a]	3950 miles
r_M radius of Moon	1350 miles	1080 miles
r_S radius of Sun	27,000 miles	432,000 miles[b]
D_M Earth to Moon	80,000 miles	238,868 miles[c]
D_S Earth to Sun	1,600,000 miles	93×10^6 miles[c]

[a] As already pointed out, this is taken from Eratosthenes. While Eratosthenes lived about 40 years after Aristarchus, similar estimates had been made earlier.

[b] The Sun consists of gas. The radius given is that of the photosphere, the illuminated part. The part from the center to 30% of its radius has in essence all the shining power and 60% of the mass. The part from the center to 60% of its radius has 95% of the mass.

[c] Radar measurements.

In any case, using only some basic observations and pure thought (i.e., mathematics), Aristarchus provided at least some idea of the magnitude of the distances in the solar system and began to unravel some of its mystery. In fact, Aristarchus's strategy is correct in principle. With more accurate measurements of the angles involved, he would have done much better; see Exercise 26. More serious is the problem of refraction. A ray of light bends as it moves from one medium to another of different density, and in particular, it bends when it moves through the atmosphere. The distance D_S was not computed with suitable accuracy until the 17th century, when the Italian astronomer Casini came within $7\frac{1}{2}\%$ of the correct value. More accurate values were not obtained until the latter part of the 18th. The calculations for these estimates used information about the observed path of Venus across the Sun.

1.6 The Sandreckoner

Archimedes—the most famous scientist of antiquity—was born around 285 B.C. in the Greek city state of Syracuse, a port on the Mediterranean island of Sicily. There is historical reference to the fact that he spent considerable time in Alexandria, and there seems little doubt that he studied with the successors of Euclid. After his studies, he returned to Syracuse and lived there in complete absorption with his mathematical investigations.

Late in the 3rd century B.C., Syracuse became embroiled in the struggle between Rome and Carthage for control of the western Mediterranean. In his *Parallel Lives*, the historian Plutarch (about 46–126 A.D.) recounts Archimedes's efforts in the defense of the city against the Romans:

> When, therefore, the Romans assaulted the walls in two places at once, fear and consternation stupefied the Syracusans, believing that nothing was able to resist that violence and those forces. But when Archimedes began to ply his engines, he at once shot against the land forces all sorts of missile weapons, and immense masses of stone that came down with incredible noise and violence, against which no man could stand; for they knocked down those upon whom they fell, in heaps, breaking all their ranks and files. In the meantime huge poles thrust out from the walls over the ships, sunk some by the great weights which they let down from on high upon them; others they lifted up into the air by an iron hand [and soon] such terror had seized upon the Romans, that, if they did but see a little rope or a piece of wood from the wall, instantly crying out, that there it was again, Archimedes was about to let fly some engine at them, they turned their backs and fled.

The Roman attack on Syracuse was repelled. A lengthy siege was later successful and Syracuse was conquered and destroyed in 212 B.C. Archimedes perished during the destruction. Plutarch relates several versions of his death. The one most widely cited finds Archimedes, oblivious to the city's capture, absorbed in the study of a particular diagram that he had sketched in the sand. When a Roman soldier confronted him, Archimedes requested time to complete his deliberations. The impatient soldier, however, ran him through with his sword.

The work of Archimedes is impressive—as we shall soon see. Plutarch speaks of Archimedes's

> purer speculations [and] studies, the superiority of which to all others is unquestioned, and in which the only doubt can be, whether the beauty and grandeur of the subjects examined, or the precision and cogency of the methods and means of proof, most deserve our admiration. It is not possible to find in all geometry more difficult and intricate questions, or more simple and lucid explanation. No amount of investigation of yours would succeed in attaining the proof, and yet once seen, you immediately believe you would have discovered it; by so smooth and so rapid a path he leads you to the conclusion required.

Archimedes was also the quintessential eccentric scientist. His deep absorption in thought

> made him forget his food and neglect his person, to that degree that when he was occasionally carried by absolute violence to bathe, or have his body anointed, he used to trace geometrical figures in the ashes of the fire, and diagrams in the oil on his body.

A famous episode recounts how, after a particularly satisfying discovery, Archimedes ran through the streets of Syracuse in naked celebration shouting "Eureka, Eureka!" (Eureka, or *ευρεκα*, is Greek for "I have found it.")

It is, of course, difficult to separate fact from fiction and reality from legend in Plutarch's account of Archimedes's remarkable talents as inventor of machines of war. The ingenuity of Archimedes's speculations about geometry and physics, however, can be corroborated. Much of his work has survived in transmitted form.

We turn now to Archimedes's scheme for writing large numbers. You will encounter the basic Greek number system in the exercises. It allows for numbers as large as

$$99{,}999{,}999 = 9999(10{,}000) + 9999 = \ ,\delta\lambda\varsigma\delta M,\overline{\delta\lambda\varsigma\delta}$$

but not larger. Archimedes enlarged the basic Greek number system into an incredible scheme. He introduced the number

$$MM = (10{,}000)(10{,}000) = 100{,}000{,}000 = 10^8.$$

and referred to numbers up to MM as numbers of the *first order*. He then built the following tower of "orders" and "periods" of numbers. In modern notation:

First order:	The numbers N with $1 \le N < 10^8 = 10^{8\cdot\mathbf{1}}$
Second order:	The numbers N with $10^{8\cdot 1} = 10^8 \le N < 10^{16} = 10^{8\cdot\mathbf{2}}$
Third order:	The numbers N with $10^{8\cdot 2} = 10^{16} \le N < 10^{24} = 10^{8\cdot\mathbf{3}}$
Fourth order:	The numbers N with $10^{8\cdot 3} = 10^{24} \le N < 10^{32} = 10^{8\cdot\mathbf{4}}$
$\vdots$	
$\mathbf{10^8}$th order:	The numbers N with $10^{8(10^8-1)} \le N < 10^{8\cdot\mathbf{10^8}}$.

These are the numbers of the **first period**. This is only the beginning. The numbers of the **second period** are also partitioned into orders:

First order:	The numbers N with $10^{8\cdot 10^8} \le N < 10^{8(10^8+\mathbf{1})}$
Second order:	The numbers N with $10^{8(10^8+1)} \le N < 10^{8(10^8+\mathbf{2})}$
Third order:	The numbers N with $10^{8(10^8+2)} \le N < 10^{8(10^8+\mathbf{3})}$

Fourth order: The numbers N with

$$10^{8(10^8+3)} \leq N < 10^{8(10^8+\mathbf{4})}$$

$$\vdots$$

$\mathbf{10^8}$th order: The numbers N with

$$10^{8(10^8+10^8-1)\cdot 8} \leq N < 10^{8(10^8+\mathbf{10^8})}.$$

The **second period** ends with (one less than) the number $10^{8(10^8+10^8)} = (10^{8\cdot 10^8})^{\mathbf{2}}$. The **third**, **fourth**, **fifth**, etc.... **periods** follow. Finally, with the $\mathbf{10^8}$**th period** and the number $(10^{8(10^8)})^{\mathbf{10^8}} = 10^{8\cdot 10^{16}}$, the array stops.

To put all of this into perspective, note that the visible universe is thought to have on the order of 10^{80} atoms (about 99.9% of them hydrogen and helium). This is Eddington's number, after the British astrophysicist Arthur Stanley Eddington. Since $10^{80} = 10^{8\cdot 10}$, this is the first number of the 11th order of the first period. The point is not so much the usefulness of Archimedes's scheme, but rather the grandiose nature of his speculations. Archimedes, in other words, thought big!

Archimedes looked for a context in which to illustrate the utility of his cosmic array of numbers. Having found it, he was evidently very pleased to address his manuscript *The Sandreckoner* to his benefactor the king of Syracuse. He began by giving the king an astronomy lesson:

> Aristarchus brought out a book consisting of some of the hypotheses, in which the premises lead to the result that the universe is many times greater than that now so called. His hypotheses are that the fixed stars and the Sun remain unmoved, that the Earth revolves about the Sun in the circumference of a circle, the Sun lying in the middle of the orbit, and that the sphere of the fixed stars, situated about the same center as the Sun, is so great ...

Then he stated his purpose:

> I say then that, even if a sphere were made up of sand, as great as Aristarchus supposes the sphere of the fixed stars to be, I shall still prove that, of the numbers named by me, some exceed in multitude the number of grains of sand in a mass which is equal in magnitude to the sphere referred to, provided that the following assumptions are made ...

In other words, Archimedes imagined the entire cosmos to be packed with sand and proposed to count the number of grains of sand in question!

Our description of Archimedes's discussion will use earlier notation: r_E for the radius of the Earth, r_M for the radius of the Moon, r_S for the radius of the Sun, and D_S for the distance between the Sun and the Earth.

What assumptions did Archimedes make? The first concerned r_E:

$$r_E \leq 47{,}500 \text{ miles.}$$

We have seen that Eratosthenes's rather accurate estimate was 3850 miles. So here Archimedes thought too big. Remember, however, that it was his purpose to display the vastness of his number scheme. Following Aristarchus, Archimedes supposed that

$$r_M < r_E.$$

Recalling that Aristarchus had shown that r_S is about 20 times greater than r_M, he next assumed that

$$r_S \leq 30 r_M.$$

This is in fact too small. A look at Table 1.4 shows, after a quick calculation, that $r_S = 400 r_M$.

Because $r_S \leq 30 r_M < 30 r_E \leq (30)(47{,}500)$, Archimedes got

$$r_S \leq 1{,}425{,}000 \text{ miles.}$$

While these inequalities are based on a combination of earlier estimates and pure speculation, Archimedes next turned to careful observation and delicate geometrical arguments. He used a long rod

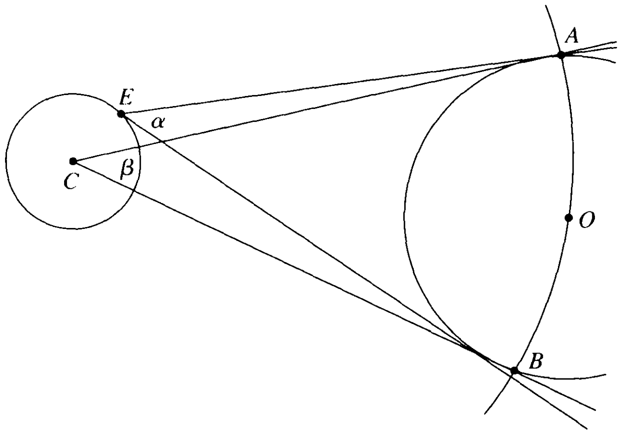


Figure 1.29

with a small disc fastened at its end. He pointed it in the direction of the Sun just after sunrise and carefully measured the angle α that the Sun subtends in the sky. He determined that $(\frac{90}{200})^\circ < \alpha < (\frac{90}{164})^\circ$ or, in radian measure,

$$\frac{1}{200}\frac{\pi}{2} < \alpha < \frac{1}{164}\frac{\pi}{2}.$$

The implied estimate $\alpha \approx \frac{1}{2}^\circ$ for the so-called angular diameter of the Sun is accurate. (Aristarchus took it to be 2°. Refer to Figure 1.24.)

Archimedes imagined an observer on the Earth looking out at the Sun just after sunrise. He constructed the diagram of Figure 1.29 and positioned the observer at the point E. Two tangent lines are drawn from E to the Sun. Note that α is the angle that these tangent lines determine at E. From C, the center of the Earth, he placed two more tangent lines to the Sun and let β be the angle that they determine. Taking the center of the Sun to be O, he obtained a circular arc by rotating CO, and he let A and B be the two points of intersection of this arc with the tangents from C. Since $\beta = \frac{\operatorname{arc} AB}{D_S}$,

$$D_S = \frac{\operatorname{arc} AB}{\beta} \approx \frac{2AO}{\beta} = \frac{2r_S}{\beta}.$$

Since the Sun is far away and E and C relatively close, observe that $\alpha \approx \beta$. (Archimedes showed much more. By a very careful argument that used delicate geometry and trigonometry, for example, a formula equivalent to $\frac{\sin\varepsilon}{\sin\gamma} < \frac{\varepsilon}{\gamma} < \frac{\tan\varepsilon}{\tan\gamma}$ for $0 < \gamma < \varepsilon < \frac{\pi}{2}$, he verified that $\beta < \alpha < \frac{100}{99}\beta$.) So Archimedes obtained the approximations

$$D_S \approx \frac{\operatorname{arc} AB}{\alpha} \approx \frac{2r_S}{\alpha}.$$

Inserting the inequalities $r_S \leq (30)(47{,}500)$ miles and $\frac{1}{200}\frac{\pi}{2} < \alpha$, he found that $D_S < \frac{(60)(47{,}500)(400)}{\pi} < 160 \times 10^6$ miles. He therefore obtained

$$D_S < 160 \times 10^6 \text{ miles.}$$

Let D_* be the distance to the stars. Speculating about another of Aristarchus's assertions, Archimedes took

$$D_* < 1.6 \times 10^{12} \text{ miles.}$$

It is a strange curiosity that these last two inequalities correspond in a sense to correct values (Table 1.5).

Archimedes could now turn to the computation of the number of grains of sand needed to fill the sphere of the universe, i.e., the sphere of radius D_*. See Figure 1.30. He began with a poppy seed and assumed that a sphere the size of a poppy seed could hold 10,000 grains of sand. (The sand that Archimedes has in mind is evidently very finely grained.) He next estimated that the diameters of 40 poppy seeds added to one "fingerbreadth." Now, the volume of a sphere of radius r is

Table 1.5

	Archimedes[a]	**Actual**
D_S Earth to Sun	160×10^6 miles	93×10^6 miles
D_* Earth to Stars	1.6×10^{12} miles	24×10^{12} miles[b]

[a] The numbers in the column are to be understood to be bigger than the distances to which they refer. For example, D_S is less than the indicated number. Note also that these numbers were based mostly on speculation. These distances were not determined with any finality until more than 2000 years later.

[b] This is the approximate distance to the nearest stars; see the discussion in Section 1.7 (Postscript).

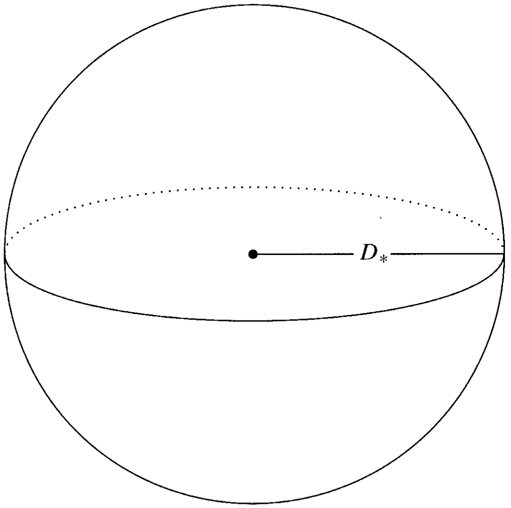


Figure 1.30

$\frac{4}{3}\pi r^3$, or $\frac{4}{3}\pi(\frac{d}{2})^3 = \frac{\pi}{6}d^3$ in terms of its diameter d. It follows that if the diameter of a sphere is increased by a factor of $2, 3, 4, \ldots, k, \ldots$, then its volume is increased by the factor 2^3, 3^3, 4^3, $\ldots$, $k^3, \ldots$. So a sphere of diameter one finger-breadth has a volume of 40^3 times greater than that of a poppy seed. Therefore, it can be filled with $10{,}000 \times 40^3 = 4^3 \times 10^3 \times 10^4 \approx 10^9$ grains of sand. Continuing in this way, Archimedes concluded that 10^{63} grains of sand will fill a sphere of radius 1.6×10^{12} miles. Since $D_* < 1.6 \times 10^{12}$ miles, this is more than enough to fill the sphere of the universe. Because Archimedes's number system handles 10^{63} with ease, his goal was accomplished!

By a remarkable coincidence, the 10^{63} grains of sand correspond to about 10^{80} atoms. So Archimedes arrived at a total amount of cosmic matter that isn't far from Eddington's 20th century estimate.

1.7 Postscript

Trigonometry—on a bigger scale yet—is used (*is* used today!) to determine the distances to the near stars. The principle is simple. Imagine yourself in a moving car looking out at the scenery. As the car moves, your perspective is constantly changing. The objects that are near will zoom past, those that are farther will move past more slowly, and the distant ones, say, the Moon, will hardly move at all. The Earth, too, is moving. As it does, it affords different perspectives on the heavens. Plotting the positions of stars carefully and regularly month after month reveals that some stars change their position in a detectable and measurable way and that others remain fixed. The greater the change in a star's position, the closer it is. The smaller the change in position the farther away it is.

To measure the distance to a near star A, proceed as follows. Consider the Earth in the two opposed positions E and E' in its orbit about the Sun. (See Figure 1.31.) When the Earth is at E, make note of the position C_1 of A against the fixed pattern of distant stars in a constellation with a telescope. Since A is near, its position in the constellation is observed to have shifted to, say, C_2 when the Earth is at E'. When the Earth is back at E, measure the angle $\theta = \angle C_1EC_2$

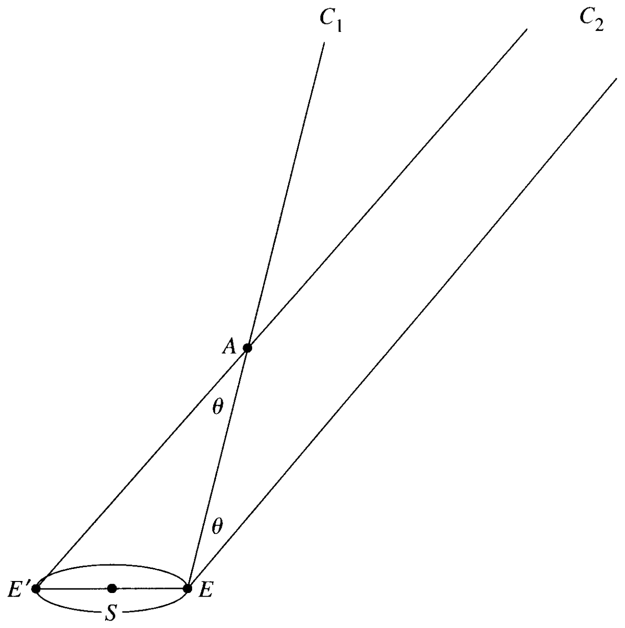


Figure 1.31

between the positions C_1 and C_2. Since the stars in the constellation are very far, the lines $E'C_2$ and EC_2 are parallel. So $\angle EAE' = \theta$.

The angle $p = \frac{1}{2}\theta$ is called the *stellar parallax* of the star A. Let D_A be the distance from the Earth to the star A. This distance is in essence the same as that from the Sun S to A. The distance between the Earth and the Sun is some 93 million miles. See Table 1.5. Astronomers refer to this distance as the *astronomical unit.* So 1 AU $= 93 \times 10^6$ miles. So the triangle ΔASE can be approximated by the circular sector shown in Figure 1.32. Letting p_{rad} be the stellar parallax of A in radian measure, it follows that

$$p_{\text{rad}} = \frac{1}{D_A},$$

where D_A is given in AUs.

Since stellar parallax is always extremely small, it is customary to measure it in seconds, where 1 second is $\frac{1}{60}$ of 1 minute, which is $\frac{1}{60}$ of 1 degree. So $1^\circ = 3600$ seconds. Therefore,

$$\begin{aligned} 1 \text{ radian} &= \left(\frac{180}{\pi}\right)^\circ \\ &= \frac{3600 \cdot 180}{\pi} \text{ seconds} = 2 \times 10^5 \text{ seconds.} \end{aligned}$$

So the parallax p is converted from radians to seconds by $p_{\text{sec}} = 2 \times 10^5 p_{\text{rad}}$.

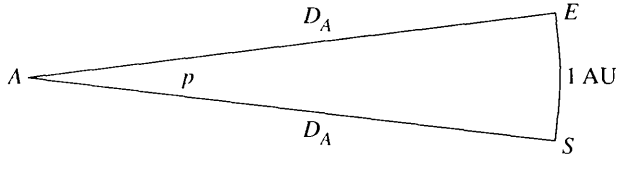


Figure 1.32

The distances to the stars are so vast that they are not measured in AUs but in *light years.* One light year, 1 LY for short, is the distance that light travels in one year. Since light travels 186,000 miles in one second, it travels approximately 6×10^{12} miles in one year. So we take 1 LY $= 6 \times 10^{12}$ miles. Since 1 AU $= 93 \times 10^6$ miles, it follows that 1 LY $= \frac{6 \times 10^{12}}{93 \times 10^6}$ AU. Therefore, 1 AU $= \frac{93 \times 10^6}{6 \times 10^{12}}$ LY $= 1.5 \times 10^{-5}$ LY. The equality $p_{\text{rad}} = \frac{1}{D_A}$ can now be converted as follows:

$$\begin{aligned} p_{\text{sec}} &= 2 \times 10^5 p_{\text{rad}} = 2 \times 10^5 \frac{1}{D_A} \text{ AU} \\ &= (2 \times 10^5)(1.5 \times 10^{-5}) \frac{1}{D_A} \text{LY} = \frac{3}{D_A} \text{LY.} \end{aligned}$$

Therefore, the distance D_A from the Earth to the star A in light years is

$$D_A = \frac{3}{p_{\text{sec}}},$$

where p_{sec} is the parallax of A measured in seconds.

The parallax of the faint star Proxima Centauri is about 0.76 seconds, or $\frac{1}{5000}^\circ$. Since this is the nearest star, its parallax is the largest. Inserting $p_{\text{sec}} = 0.76$ into the preceding equation shows that the distance to Proxima Centauri is approximately $\frac{3}{0.76} = 4$ LY. Since 1 LY $= 6 \times 10^{12}$ miles, Proxima Centauri is about 24×10^{12} miles distant (see Table 1.5). Stellar parallaxes were measured with good accuracy in the 19th century. For example, Wilhelm Bessel, the German mathematician and astronomer, measured the stellar parallax of 61 Cygni in 1838 and came within 10% of the modern value of 0.27 seconds. Popping $p_{\text{sec}} = 0.27$ into the equation just derived, we find that 61 Cygni is 10.9 LY away. Sirius, the brightest star, has a parallax of about 0.38 seconds. This yields a distance of about 8 LY.

Modern methods for measuring distances to the more distant stars and galaxies include analyzing the convergence or divergence of the motion of individual stars in a cluster to determine the distance of the cluster; and analyzing the light of a particular star to determine its intrinsic luminosity, which, in combination with its brightness as measured from Earth, gives an indication of its distance. The size of the universe is, of course, considerably bigger than even Archimedes speculated. Some galaxies are millions of LYs distant and some have a size of hundreds of thousands of LYs.

Exercises

1A. The Greek Number System

One version (there are variations) of the traditional number system of the Greeks is the following:

Units	Tens	Hundreds
$\alpha = 1$ (alpha)	$\iota = 10$ (iota)	$\rho = 100$ (rho)
$\beta = 2$ (beta)	$\kappa = 20$ (kappa)	$\sigma = 200$ (sigma)
$\gamma = 3$ (gamma)	$\lambda = 30$ (lambda)	$\tau = 300$ (tau)
$\delta = 4$ (delta)	$\mu = 40$ (mu)	$\upsilon = 400$ (upsilon)
$\varepsilon = 5$ (epsilon)	$\nu = 50$ (nu)	$\phi = 500$ (phi)
ϛ $= 6$ (digamma)	$\xi = 60$ (xi)	$\chi = 600$ (chi)
$\zeta = 7$ (zeta)	$o = 70$ (omicron)	$\psi = 700$ (psi)
$\eta = 8$ (eta)	$\pi = 80$ (pi)	$\omega = 800$ (omega)
$\theta = 9$ (theta)	ϟ $= 90$ (koppa)	ϡ $= 900$ (sampi)

(The digamma, koppa, and sampi were taken from an older alphabet of the Phoenicians.)

Other numbers are formed by juxtaposition, using the rule that larger numbers go on the left and smaller ones on the right. For example: $\kappa\varepsilon = 25$, $\lambda\zeta = 37$, $\upsilon\pi\eta = 488$. To designate thousands, the units symbols were used with a stroke before the letter to avoid confusion. For example: $,\gamma = 3000$ and $,\beta\tau\pi\delta = 2384$. (To distinguish between numerals and letters, the Greeks sometimes put a bar over the numerals: $\overline{,\varepsilon\chi o} = 5670$.) For 10,000 an M (for $\mu\upsilon\rho\iota\alpha\sigma =$ myriad) was used. This was combined with other symbols as follows:

$$\beta\mathrm{M} = 20{,}000,$$

$$\iota\delta\mathrm{M},\eta\phi\xi\zeta = 14(10{,}000) + 8567 = 148{,}567$$

$$\upsilon\mathrm{M} = (400)(10{,}000) = 4{,}000{,}000$$

$$\omega\mu\varepsilon\mathrm{M},\beta\tau\pi\delta = 845(10{,}000) + 2384 = 8{,}452{,}384$$

Addition and multiplication were cumbersome; this is probably one reason why Greek algebra lagged behind Greek geometry.

1. Write the numbers 85; 842; 34,547; 2,875,739 using the Greek system.

1B. Greek Algebra

Problems (2)-(5) are taken from *The Greek Anthology*.[8]

2. The Muses stole and divided among themselves, in different proportions, the apples I was bringing from Helicon. Clio got the fifth part, and Euterpe the twelfth, but divine Thalia the eighth. Melpomene carried off the twentieth part, and Terpsichore the fourth, and Erato the seventh; Polyhymnia robbed me of thirty apples, and Urania of a hundred and twenty, and Calliope went off with a load of three hundred apples. So I come to thee with lighter hands, bringing these fifty apples that the goddesses left me. How many apples did I bring from Helicon?
3. Make me a crown weighing sixty minae, mixing gold and brass, and with them tin and much wrought iron. Let the gold and brass together form two-thirds, the gold and tin together three-fourths, and the gold and iron three-fifths. Tell me how much gold you must put in, how much brass, how much tin, and how much iron, so as to make the whole crown weigh sixty minae. [A number of references describe the mina as a unit of weight roughly equal to one pound. So it seems that this crown was intended for no ordinary mortal.]
4. Throw me in, silversmith, besides the bowl itself, the third of its weight, and the fourth, and the twelfth; and casting them into the furnace stir them, and mixing them all up take out, please, the mass, and let it weigh one mina. [The first thing is to decide what the question is.]
5. Brick-maker, I am in a great hurry to erect this house. Today is cloudless, and I do not require many more bricks, but I have all I want but three hundred. Thou alone in one day couldst make as many, but thy son left off working when he had finished two hundred, and thy son-in-law when he had made two hundred and fifty. Working all together, in how many hours can you make these? [Hint: If there is not enough information, supply it.]

1C. The Quadratic Formula

6. Consider the quadratic polynomial $x^2 - 5x + 4$. Take the x coefficient -5; divide it by 2 to get $-\frac{5}{2}$; squaring gives $\left(-\frac{5}{2}\right)^2 = \left(\frac{5}{2}\right)^2$. Now rewrite $x^2 - 5x + 4$ as follows:

$$x^2 - 5x + 4 = x^2 - 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 4.$$

Check that $x^2 - 5x + \left(\frac{5}{2}\right)^2 = \left(x - \frac{5}{2}\right)^2$. Therefore

$$\begin{aligned} x^2 - 5x + 4 &= \left(x - \frac{5}{2}\right)^2 + 4 - \left(\frac{5}{2}\right)^2 \\ &= \left(x - \frac{5}{2}\right)^2 + \frac{16}{4} - \frac{25}{4} \end{aligned}$$

$$= \left(x - \frac{5}{2}\right)^2 - \frac{9}{4}.$$

We have done what is called "completing the square" for the polynomial $x^2 - 5x + 4$. Now answer the following:

- **i.** For which values of x is $x^2 - 5x + 4 = 0$?
- **ii.** What is the least value that $x^2 - 5x + 4$ can have?

7. Solve the equation $3x^2 + 21x + 12 = 0$ by completing the square for the polynomial $x^2 + 7x + 4$.

8. Let a, b, and c be constants and consider the equation $ax^2 + bx + c = 0$. What is x equal to if $a = 0$? If $a \neq 0$, use the strategy of Problems 6 and 7 to show that the solutions are given by the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1D. Rational and Real Numbers

9. The matter of measurability is perhaps most concretely illustrated by the consideration of the numbers that underlie our monetary system. All dollar amounts are expressed in the form: $\$x\frac{yz}{100}$. So only rational numbers, indeed only certain rational numbers, are allowed. In a supermarket one will occasionally find, say, 3 items for a dollar. A single item is not measurable within the system. Why not?

10. Consider the numbers 1.333333..., 2.676767..., and 4.728728728.... Show that they are rational numbers. [Hint: Let r be the number. In the first case, consider $r - 10r$.]

11. What are the decimal expansions of the rational numbers $\frac{5}{4}$ and $\frac{468}{198}$? [Hint: Carry out the divisions.]

Note: It turns out that a real number r is rational precisely if its decimal expansion has a repetitive pattern after some point. For example,

$$234.5\underline{9}99999999\ldots = 234.6 = 234\frac{6}{10}$$

and

$$52.\underline{36}36\ldots = 52\frac{468}{198}$$

are rational numbers. So is

$$35.346726\underline{38}638638\ldots.$$

12. Recall that a prime number is a positive integer $p > 1$ that has no divisors except 1 and p. Euclid pursued the study of prime numbers in Book 9 of the *Elements*. The following fact about prime numbers is known as the Fundamental Theorem of Arithmetic: Every positive integer n is a product $n = p_1^{k_1} \cdots p_i^{k_i}$ of powers of distinct prime numbers $p_1, \ldots, p_i$; and there is only one way of doing this (aside from rearranging the order of the factors). For example, $54 = 2 \cdot 3 \cdot 3 \cdot 3 = 2^1 \cdot 3^3 = 2 \cdot 3^3$ is the unique factorization of 54 into prime powers.

- **i.** Determine the factorizations of 28, 192, and 143 into powers of distinct primes.
- **ii.** Use the Fundamental Theorem of Arithmetic to show that $\sqrt{3}$ is irrational.
- **iii.** Show that a positive integer n is a square precisely when all the exponents of its factorization into primes are even integers.
- **iv.** Let n be a positive integer. Use the Fundamental Theorem of Arithmetic to show that $\sqrt{n}$ is rational only if n is square.

1E. Angles and Circular Arcs

13. Fill in the following blanks.

- **i.** 1 radian = ________ degrees
- **ii.** 1 degree = ________ radians
- **iii.** 78.5° = ________ radians
- **iv.** 1.238 radians = ________ degrees

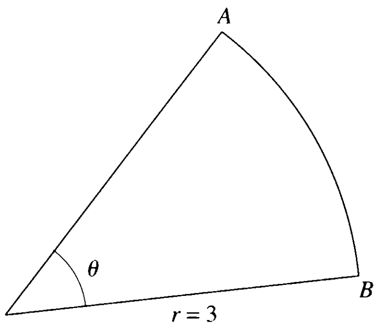


Figure 1.33

14. In the circular sector of Figure 1.33, $\theta = 57.3^\circ$. What is the length of the arc AB?

15. Start with Figure 1.13 of the text. Take the tangent to the circle at A and let P' be its point of intersection with $A'P$. By Figure 1.34, it is plausible that arc $AP \leq AP' + P'P$. Show that $AP' < A'P'$. Conclude that the

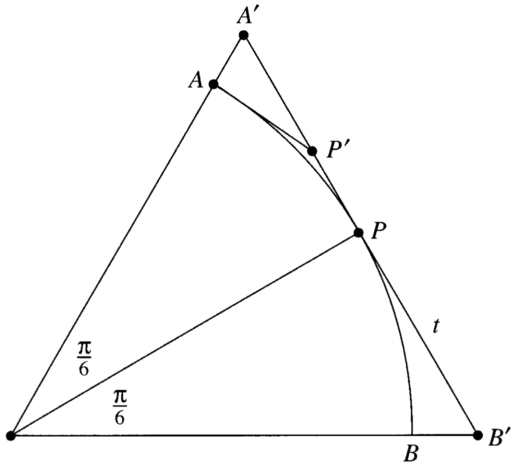


Figure 1.34

inequalities arc $AP < A'P = t$ and arc $AB < A'B' = 2t$ are plausible.

16. A rock is hurled with a sling. Just before it flies off, it is twirled in a circular arc of radius 3 feet at 4 revolutions per second. With what speed (in feet per second) does the rock fly off?

17. The circle in Figure 1.35 has center O and a radius of 2 feet. The arrow is rotating clockwise at a rate of one revolution in twelve hours (so think of it as the hour hand of a clock). The points A and D are positioned in such a way that the segment connecting them is parallel to the diameter BC of the circle. Determine the length of the arc CD, given that the arrow requires 7.5 hours to rotate from A to D.

1F. Basic Trigonometry

18. Use the appropriate triangles to fill in the values:

i. $\cos\frac{\pi}{6} =$ ________ **ii.** $\cos\frac{\pi}{4} =$ ________
iii. $\cos\frac{\pi}{3} =$ ________ **iv.** $\tan\frac{\pi}{6} =$ ________
v. $\tan\frac{\pi}{4} =$ ________ **vi.** $\tan\frac{\pi}{3} =$ ________

19. Use Figure 1.21 of the text to determine the following limits:

i. $\lim_{\theta\to 0}\cos\theta =$ ________
ii. $\lim_{\phi\to\frac{\pi}{2}}\cos\theta =$ ________
iii. $\lim_{\theta\to 0}\tan\theta =$ ________
iv. $\lim_{\phi\to\frac{\pi}{2}}\tan\theta =$ ________

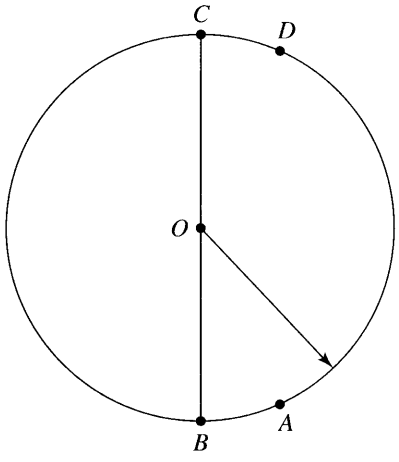


Figure 1.35

20. Illustrate with a diagram that if $\theta' > \theta > 0$, then $\cos\theta' < \cos\theta$ and $\tan\theta < \tan\theta'$.

21. The secant of θ is defined by $\sec\theta = \frac{1}{\cos\theta}$. Verify the identity $\sec^2\theta = \tan^2\theta + 1$.

22. Compare (use a calculator) the values of α, $\sin\alpha$, $\tan\alpha$ for

i. $\alpha = 0.1$ radians:
$\sin\alpha =$ ________, $\tan\alpha =$ ________.
ii. $\alpha = 0.01$ radians:
$\sin\alpha =$ ________, $\tan\alpha =$ ________.
iii. $\alpha = 0.001$ radians:
$\sin\alpha =$ ________, $\tan\alpha =$ ________.

1G. Distances and Sizes in the Universe

23. Compute r_M, r_S, D_M, and D_S using Aristarchus's argument and

a. Keep $r_E = 3850$ miles.
b. In hypothesis **C**, take $89°50'$ instead of $87°$. (The angle measure $'$ is called minute; $1'$ is equal to $\frac{1}{60}^{\circ}$.)
c. In hypothesis **D**, take $\frac{1}{2}^{\circ}$ instead of $2°$. (So the angle in Figure 1.27 is $\frac{1}{4}^{\circ}$ instead of $1°$.)
d. In hypothesis **E**, take $5r_M$ instead of $4r_M$.

Round off to get an accuracy up to 4 decimal places. Compare your conclusions with the modern values from Table 1.4.

24. Both Aristarchus and Archimedes assumed that the Earth is a sphere. Is this reasonable in view of the mountain ranges on its surface? You are given that the radius r_E of the Earth is 3950 miles, that the height of Mount Everest is 29,028 feet, that 1 mile = 5280 feet,[9] and that the radius of a basketball is 4.7 inches. If the Earth were shrunk to the size of a basketball, how high would Mount Everest be? Is this higher than one of the little mounds—officially called a pebble—on a basketball? These have a height of about 0.02 inches.[10]

25. Since 3950 miles $= 20,856,000$ feet and 4.7 inches = 0.39 feet, the shrinking factor in Problem 24 is $\frac{0.39}{20{,}856{,}000}$, or about $\frac{1}{50{,}000{,}000}$. Show that if the radius r_M of the Moon, the distance D_M to the Moon, the radius r_S of the Sun, the distance D_S to the Sun, and the distance D_* to the nearest star were shrunk by this factor, we would have (approximately):

$r_E = 4.7$ inches (the radius of a basketball, as we have already seen)

$r_M = 1.37$ inches (about the radius of a baseball which is 1.43 inches)

$D_M = 25$ feet

$r_S = 45$ feet

$D_S = 1.86$ miles

$D_* = \frac{1}{2}$ million miles

26. Shrink the Sun to the size of a basketball. What is the shrinking factor? Shrink the rest of the solar system by this factor and compute r_E, r_M, D_M, r_S, D_S, and D_* in this case.

27. Reconstruct Archimedes's argument that 10^{63} grains of sand more than fill the sphere of the universe. Assume that 1 finger breadth is $\frac{2}{3}$ of an inch.

28. The near stars Barnard, G51-15, and Ross 780 have stellar parallaxes of 0.55, 0.27, and 0.21 seconds, respectively. Determine their distances in light years.

Notes

[1] This theorem was probably already known to the Babylonians in 1700 B.C., more than 1000 years before the time of Pythagoras (about 570 to 500 B.C.).

[2] The Pythagorean philosophy was not without later influence, however. The book by G.L. Hershey, *Pythagorean Palaces* (Cornell University Press, Ithaca N.Y. and London, 1976), was written to establish the fact that in "the Italian Rennaissance domestic architecture was largely ruled by Pythagorean principles."

[3] This is a question that occupied the mathematicians of antiquity and is still relevant today. Our discussion will give only a very elementary perspective.

[4] When a segment, such as $A'P$ or PB', appears in a mathematical expression, the reference here and elsewhere in this text will always be to its length.

[5] A gnomon is simply a straight stick or rod. The word comes from gno, "to know," in ancient Greek. Our words prognosis and physiognomy are derived from it.

[6] The Greeks used stadia, not miles. Ten stadia are the equivalent of about one mile.

[7] The detail on the Earth in the figures that follow should not mislead the reader into thinking that the Greeks of the third century B.C. understood the scope and shape of Africa, Northern Europe, and the Atlantic Ocean. The Greeks were very familiar with the territory near the Mediterranean Sea and the armies of Alexander the Great advanced all the way to today's India and Afghanistan. However, a true concept of the extent of the continental land masses and oceans began to develop only with the voyages of discovery in the 15th century.

[8] *The Greek Anthology* is a collection of Greek poems, songs, and riddles. Some of these were compiled as early as the 7th century B.C. and others as late as the 10th century A.D. Harvard University Press produced a new edition of *The Greek Anthology* in 1993.

[9] So Mount Everest is about 5.5 miles high. As an aside, note the Earth is actually not quite a sphere. The Earth's rotation has caused an "equatorial bulge," so that the Earth's diameter through the equator is about 26 miles more than that through the poles.

[10] According to Rawlings Sporting Goods, the official circumference of a basketball is from 29.5 to 30 inches. This translates to a radius r of 4.695 to 4.775 inches. A pebble has an official height from 0.013 to 0.025 inches.

2

Ptolemy and the Dynamics of the Universe

So far we have considered the "static" situation of the universe: the sizes of the Earth, Moon, and Sun, and the various distances between them. Next we turn to the question of its "dynamics": according to what schemes do the heavenly bodies move?

2.1 A Geometry of Shadows and the Motion of the Sun

The Greek philosophers perceived the basic structure of the universe to be this: The Earth is spherical in shape, it is fixed, and it lies at the center of the universe. The Moon, Mercury, Venus, Sun, Mars, Jupiter, and Saturn are the planets. (Neptune, Uranus, and Pluto were not yet known.) All is surrounded by the *celestial sphere* of fixed stars, which rotates about its axis once a day. The motion of the planets (including the Sun) has two components: they are carried along by the daily rotation of the sphere of stars, but they also have their own independent motion. The planets, in particular, are observed to change their position relative to the background of fixed stars. Each of the Greek thinkers provided additional details. For example, according to Aristotle, each of the planets and the Sun is attached to its own crystal sphere whose rotation moves it along; and according to Pythagoras, each of the heavenly bodies contributes a certain tone to an orchestrated "music of the spheres." The reasons given in support of this scheme included, "We must suppose the Earth, the Hearth of the House of the gods, to remain fixed and the planets with the whole embracing heavens to move" and "The motion must be circular because circles and spheres are the only perfect shapes." The Sun-centered model of Aristarchus was dismissed as being in obvious contradiction to the facts.

The essence of the structure of the Greek universe is illustrated in Figure 2.1. The axis is the continuation of the north-south axis of the Earth. The *celestial equator* is the projection of the Earth's equator against the celestial sphere. The *ecliptic* is the projection of the path of the Sun against the celestial sphere. Both the celestial equator and the ecliptic are great circles on the celestial sphere. The two points where these circles intersect are called the *vernal* and *autumnal equinox* positions. The angle ε between the planes determined by the celestial equator and the ecliptic is called the *obliquity of the ecliptic*. The point on the ecliptic where the Sun reaches its highest position above the plane of the celestial equator is the *summer solstice* position. The point of lowest position below this plane is the *winter solstice* position. Note that this terminology is still in use today.

Today we know that the Earth rotates once a day about its north-south axis and that it moves in a year-long orbit about the Sun. However, from the vantage point of an observer on the surface of the Earth—and this was the point of view of the Greek philosophers—the daily rotation of the Earth is seen as the daily rotation of the celestial sphere of stars. The motion of the Earth in its orbit around the Sun is seen as that of the Sun against the celestial sphere in a yearly cycle. Our attention turns to the observations by the Greeks of this apparent motion of the Sun.

The following ritual must have been performed by the Greek astronomers thousands of times. Plant a gnomon—this is the pole Eratosthenes used in his measurement of the Earth—at a point A in the middle of a flat horizontal stretch of ground. Use a plumb line to set it in vertical position GA. Refer to Figure 2.2. At some time in the morning, after the Sun has risen in the east, mark the endpoint W of the shadow (it lies west of A) and measure the length AW. As the Sun moves up in the sky, the shadow will shorten and rotate. When the Sun starts its descent, the shadow will begin to lengthen. At some time before the Sun sets in the west, the tip of the shadow will reach a point E (toward the east) on the ground where its length AE will equal AW. By the symmetry of the process, the bisector of the angle WAE lies on the north-south line through A. This line is called the *meridian* through A. At the moment in the day when the shadow falls on the meridian, it has reached its shortest length; the Sun is now at its highest point in the sky. This moment is *noon* on that day. The length of a day is defined to be

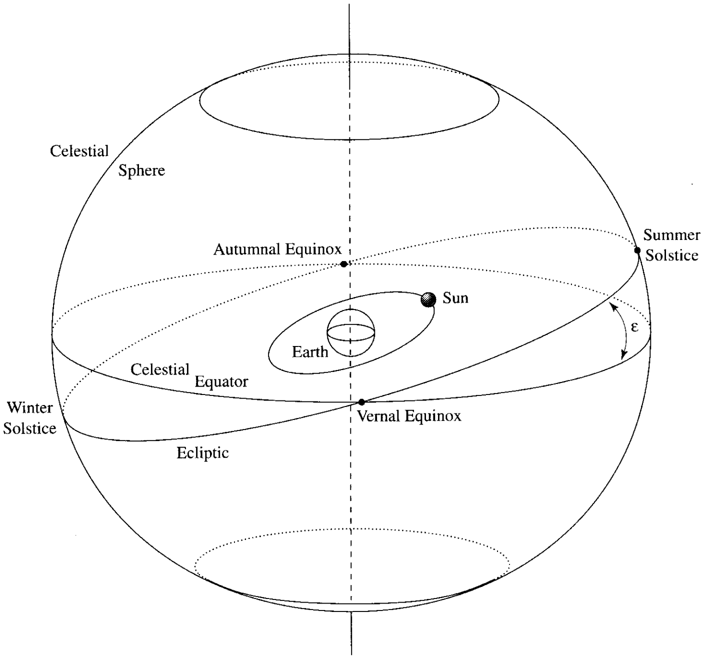


Figure 2.1

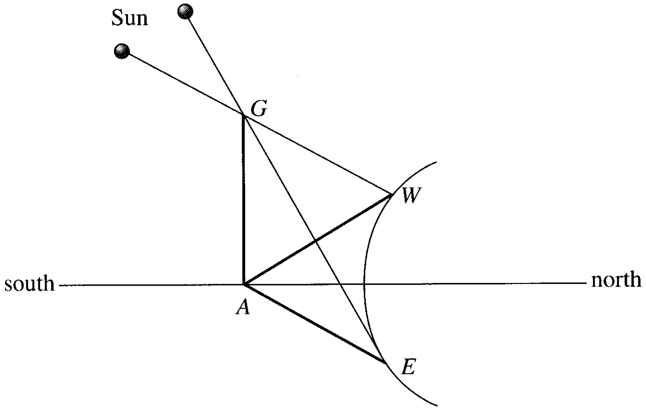


Figure 2.2

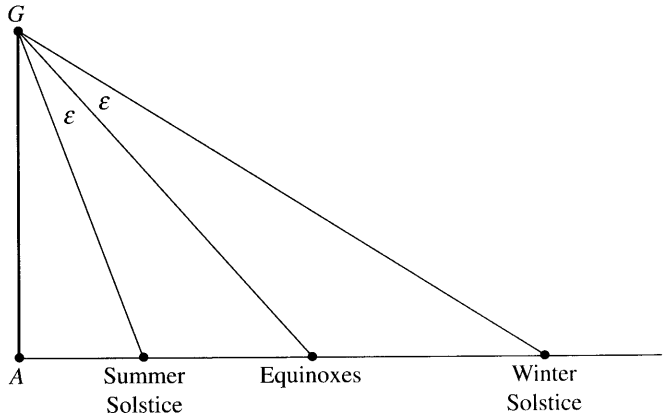


Figure 2.3

the duration of time between two successive noons. The division of a day into 24 equal units defines the hour. The hour, in turn, is divided into 60 equal units called minutes. (Note the influence of the Babylonian base 60 number system.)

Since the focus is on the movement of the Sun against the celestial sphere, the daily rotation of this sphere (in other words, the daily rotation of the Earth) must be discounted. To achieve this, gnomon measurements are taken exactly at noon each day. The day on which the noon shadow of the gnomon is shortest, in other words, the day of the year on which the Sun is at its very highest point in the sky, is summer solstice. Winter solstice is the day on which the noon shadow of the gnomon is longest. This is the day of the year on which the noon Sun is at its lowest point. Summer solstice is the day of longest daylight, and winter solstice is the day of shortest daylight. There are two days on which day and night have the same length. The one occurring before summer solstice is the vernal equinox, and the other, occurring after summer solstice, is the autumnal equinox. Some thought (and reference to Figure 2.1) shows that this occurs when the Sun is in the plane of the celestial equator. This corresponds to the shadow position determined by the bisection of the angle given by the shadow's summer and winter solstice positions: see Figure 2.3. The angle ε obtained by the bisection is the obliquity of the ecliptic.

The seasons of the year are determined as follows: *Spring* is the period from vernal equinox to summer solstice; *summer* runs from summer solstice to autumnal equinox; *autumn* from autumnal equinox to winter solstice; and *winter* from winter solstice to vernal equinox. The year is defined to be the time period between two successive summer solstices (or winter solstices, or autumnal equinoxes...).

The astronomer-mathematician Hipparchus lived and worked from about 190–125 B.C. in Nicaea (50 miles southeast of Constantinople) and also on the island of Rhodes. His shadow measurements gave the following results:

Spring	$94\frac{1}{2}$ days
Summer	$92\frac{1}{2}$ days
Autumn	$88\frac{1}{8}$ days
Winter	$90\frac{1}{8}$ days

By addition, Hipparchus obtained that the year consists of $365\frac{1}{4}$ days. He measured ε to be about 24°.

Refer back to Figure 2.1 and observe that the vernal equinox, summer solstice, autumnal equinox, and winter solstice positions divide the ecliptic into four equal circular arcs. The fact that the lengths of the seasons are different means that the Sun traces out these four arcs in different lengths of time. This, however, stands in clear contradiction to the two fundamental principles of Greek philosophy:

(1) The Earth is fixed and positioned at the center of the universe (the geocentric hypothesis).
(2) All other bodies in the universe move with constant velocity along circular paths (which have the Earth at their center).

Another phenomenon stood in conflict with these principles: while the general pattern of motion of the planets (against the backdrop of the stars) was from west to east, they were observed to stop on occasion and reverse direction. This is the phenomenon of the planets' "retrograde motion." These difficulties appear to have moved Plato (in the *Dialogue Timaeus*) to exhort the philosophers of his day to do the apparently impossible: to develop a theory of motion for the universe that adhered to the fundamental principles (1) and (2) *and* was consistent with observed reality.

This is precisely what Claudius Ptolemy (about 100–165 A.D.) of Alexandria set out to do. His "Ptolemaic System of the Universe" is developed in his great work, the *Almagest*, one of the most influential texts in the history of science. It is an encyclopedic compilation—a grand synthesis—of Greek mathematics and astronomy. The title *Almagest* has a later origin. It comes from the Greek word *megiste* ($\mu\varepsilon\gamma\iota\sigma\tau\eta$), meaning "greatest" (probably from references to the work as the "Great Collection" or "Great Astronomer") to which Islamic astronomers added the Arabic prefix *al* (the).

2.2 Geometry in the *Almagest*

The *Almagest* contains the first surviving comprehensive exposition of Greek trigonometry (and is thus also of importance in the history of mathematics). Presupposing a thorough knowledge of the *Elements* of Euclid, Ptolemy includes careful expositions of specialized foundational results—for example, those of Apollonius (about 270–190 B.C.) of Perga (an ancient city that was located on the south coast of today's Turkey), those of Hipparchus, as well as his own. The trigonometry in the *Almagest* is primarily a science of numerical calculations—often very intricate—of triangles in the plane and on the sphere. This emphasis on numerical methods goes counter to the commonly held notion that Greek mathematics deliberately avoided such methods in favor of purely geometrical ones. Ptolemy, however, gives no indication of the rules and techniques underlying his calculations, and it is indeed true that the Greeks were reluctant to deal with such matters in the form of formal treatises.

From the large body of mathematics in the *Almagest*, we will consider only a few (very few) facts from plane geometry. This suffices for the purposes of this text.

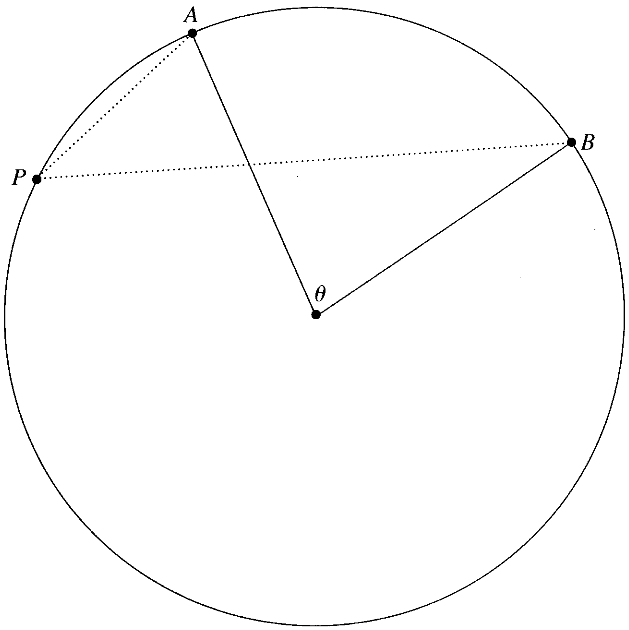


Figure 2.4

Proposition 2.1. Let A and B be any two points on the circle in Figure 2.4 and let θ be the angle that they determine along with the center. Let P be any other point on the circle. Then

$$\angle APB = \frac{\theta}{2}.$$

One way to verify this is as follows. Consider first the special case shown in Figure 2.5, where the segment PB is a diagonal. Let $\angle APB = \alpha$. Notice that the triangle formed by A, P, and the center is isosceles and, in particular, that the angle at A is also equal to α. Adding the three angles of this triangle shows that $2\alpha + (\pi - \theta) = \pi$. Therefore, $2\alpha = \theta$, and hence $\angle APB = \alpha = \frac{\theta}{2}$, as required.

Now let's return to the general case. Draw a diameter PC from P through the center and let φ be the indicated angle. See Figure 2.6. By the special case already verified,

$$\angle APC = \frac{1}{2}(\theta + \varphi) \quad \text{and} \quad \angle BPC = \frac{1}{2}\varphi.$$

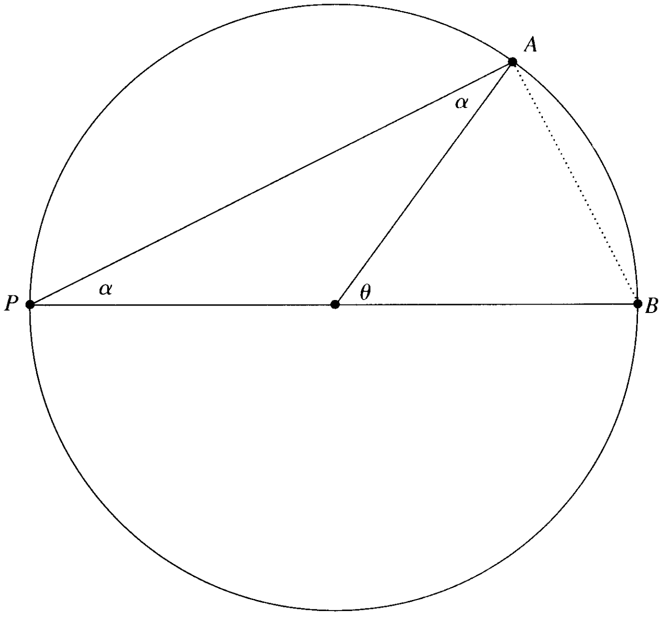


Figure 2.5

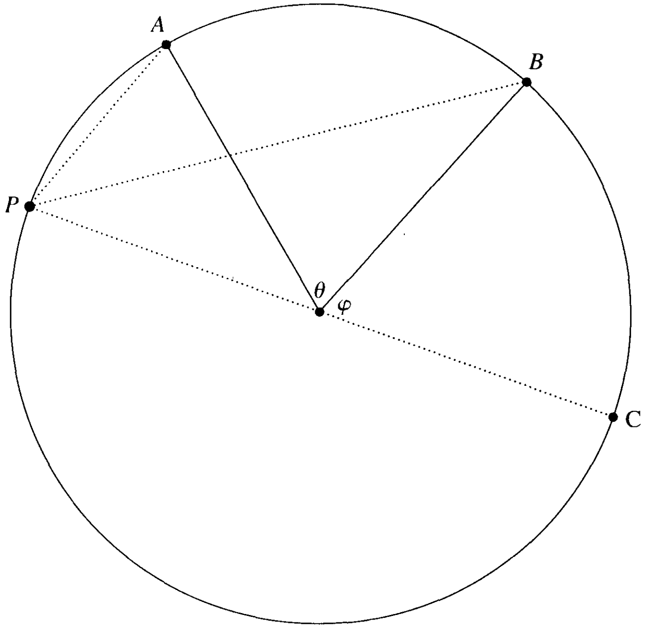


Figure 2.6

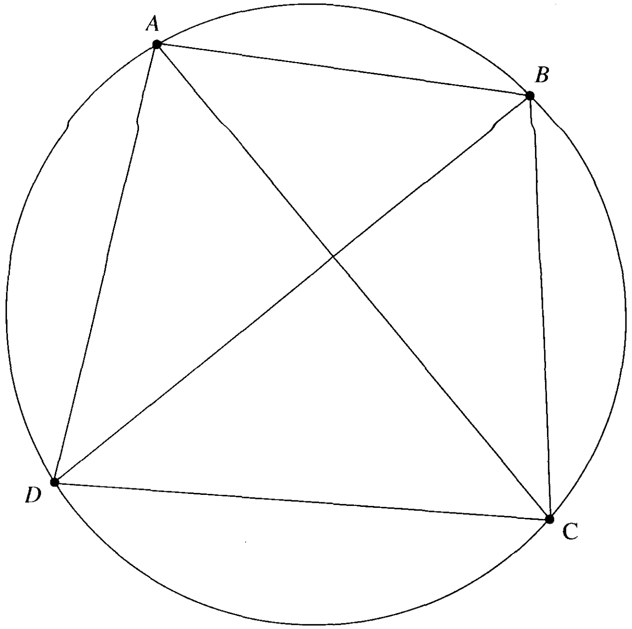


Figure 2.7

Since $\angle APB = \angle APC - \angle BPC = \frac{1}{2}(\theta + \varphi) - \frac{1}{2}\varphi = \frac{\theta}{2}$, the proof of Proposition 2.1 is complete.

Proposition 2.2. Fix A and B on the circle as in Figure 2.4. Then $\angle APB$ is the same, no matter what point P on the circle is taken.

This is so because for any point P, the angle $\angle APB$ is equal to one-half of the central angle determined by A and B.

It is now possible to derive the following famous theorem.

Ptolemy's Theorem. Let A, B, C, and D be any four points on a circle. Then

$$(AC) \cdot (BD) = (AB) \cdot (CD) + (AD) \cdot (BC).$$

Figure 2.7 shows the typical situation. Pythagoras's theorem is a special case of Ptolemy's theorem. This can be seen as follows. Complete any right triangle to a rectangle and inscribe the rectangle into a circle. Ptolemy's theorem applied to the four vertices of the triangle provides Pythagoras's theorem. Exercise 1 asks you to provide the details. Ptolemy's theorem can also be used to prove some of the basic identities of trigonometry. These include $\sin(\alpha + \beta) = (\sin \alpha)(\cos \beta) + (\cos \alpha)(\sin \beta)$ and the analogous formula for the cosine. However, there are more efficient ways to verify these identities. See Exercise 6. The proof of Ptolemy's theorem follows below. It relies on Proposition 2.2 and basic properties of similar triangles. Since it will not play a role in this book, little is lost by skipping it.

Proof of Ptolemy's Theorem. Let E be the indicated point of intersection in Figure 2.8 and draw AF such that $\angle DAF = \angle BAE$. By Proposition 2.2, $\angle ADB = \angle ACB$. So $\angle DAF = \angle BAE = \angle BAC$ and $\angle ADF = \angle ADB = \angle ACB$. Now refer to the triangles ΔADF and ΔACB of Figure 2.9. Note that these triangles have two angles in common (labeled 1 and 2). Since the sum of the angles of any triangle is 180°, the angles labeled 3 are also equal. It follows that

ΔADF and ΔACB are similar. Take one more look at Figure 2.9. Since AD and AC, and DF and BC, are corresponding pairs of sides,

$$\frac{AD}{AC} = \frac{DF}{BC}. \tag{2.1}$$

By another application of Proposition 2.2 to Figure 2.8, $\angle ACD = \angle ABD = \angle ABF$. Since

$$\angle DAC = \angle DAF + \angle FAE = \angle BAE + \angle EAF = \angle BAF$$

the triangles ΔACD and ΔABF have two angles in common. They are the angles labeled 1 and 2 in Figure 2.10. Therefore, they too are similar. Since ratios of corresponding sides are equal, we see that

$$\frac{AB}{AC} = \frac{BF}{CD}. \tag{2.2}$$

By equation (2.1), $DF = \frac{AD}{AC}BC$, and by equation (2.2), $BF = \frac{AB}{AC}CD$. By a look at Figure 2.8,

$$BD = BF + DF = \frac{AB}{AC}CD + \frac{AD}{AC}BC.$$

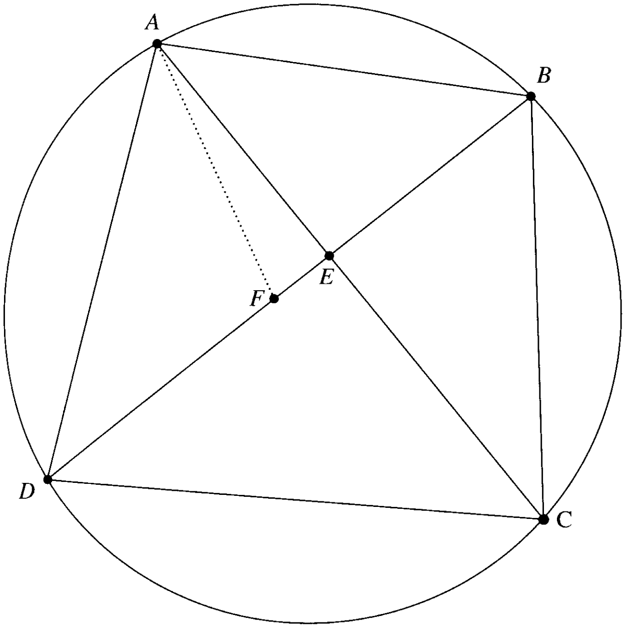


Figure 2.8

Multiplying through by AC finally shows that

$$(AC)\cdot(BD) = (AB)\cdot(CD) + (AD)\cdot(BC).$$

The proof of Ptolemy's theorem is now complete.

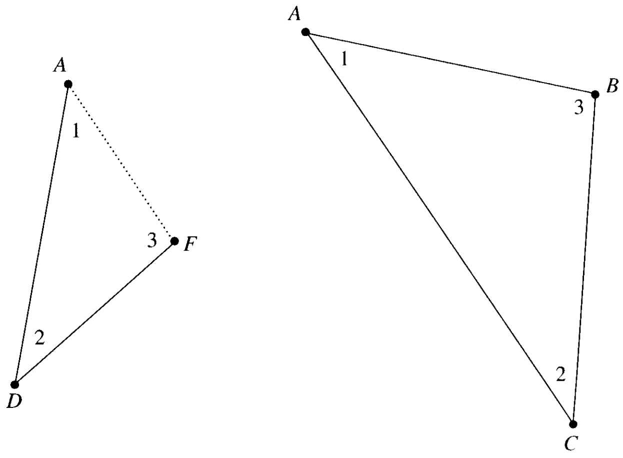


Figure 2.9

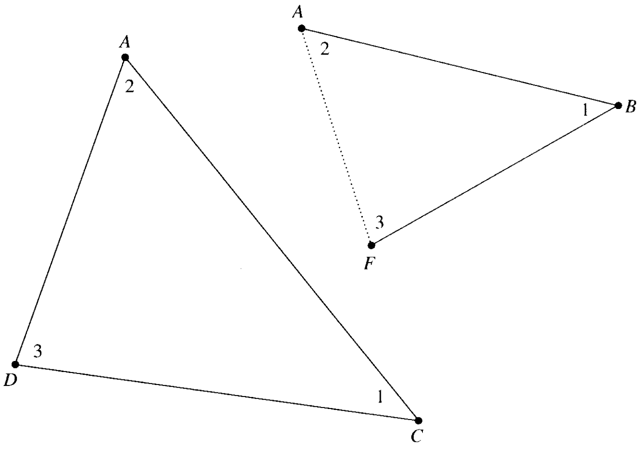


Figure 2.10

2.3 The Solar Model

Since the Sun affects everything else (consider, from the modern point of view, the fact that all the planets are in orbit around it), Ptolemy began his astronomy with it. As already pointed out, he was confronted with the following basic question: Is it possible to develop a theory that has the Sun moving with constant velocity in a circular path around the Earth and that at the same time explains the observed differences in the lengths of the seasons?

Ptolemy's solution, in principle, is this: retain the Earth at the center of the celestial sphere; retain the Sun's circular orbit and retain its constant orbital velocity; but move the center of this orbit away from the Earth! While this does compromise the fundamental principles of Greek philosophy in part, it does retain the essential elements.

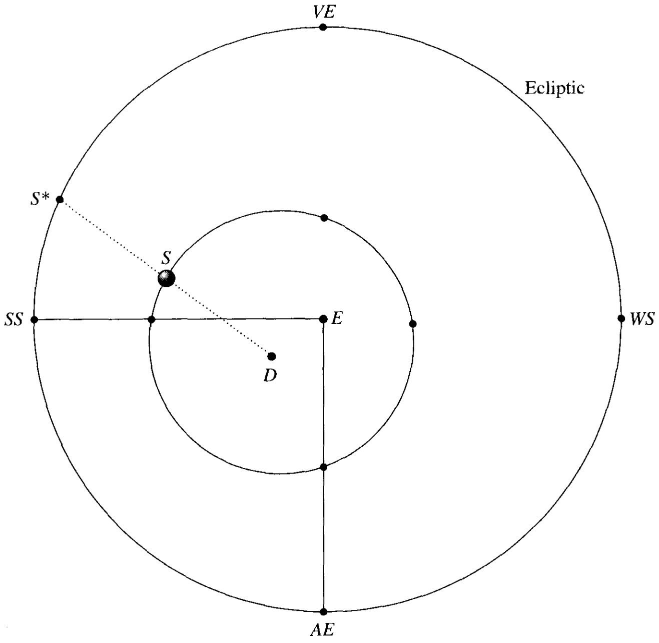


Figure 2.11

The details of the solution follow. This part of the *Almagest* is taken from the work of Hipparchus. Consider Figure 2.11. The smaller circle represents the path of the Sun; it is centered at the point D. The larger circle represents the ecliptic. Recall that this is the projection of the Sun's path on the celestial sphere. The Earth E is at the center of the ecliptic. The vernal equinox, summer solstice, autumnal equinox, and winter solstice positions are indicated by VE, SS, AE, and WS. Refer back to Figure 2.1. The Sun S moves along its circular path at constant speed. Its projection, S^*, moves on the ecliptic. Compare the motion of the Sun S to that of S^* and notice that S^* will require more time to move from SS to AE than from AE to WS, or from WS to VE, or from VE to SS. In other words, according to this scheme, summer is the longest season!

According to Hipparchus's observations, however, spring—and not summer—is the longest season. This means that in the model of Figure 2.11, D is not in the correct position. But it should now be apparent that it will be possible to keep the Earth at the center of the universe, to retain uniform circular orbits, and at the same time to allow for the differences in the lengths of the seasons.

So where exactly does D have to be placed? The answer to this question is the essence of the solar model of the *Almagest*. Figure 2.12 shows the ecliptic and the path of the Sun. The points E, D, S, and S^* continue to have the same meaning. In order to make spring the longest season, D is now placed somewhere above and to the left of E. The rest of the notation is as follows:

J is the position of the Sun in its orbit at the moment of vernal equinox.
G is the position of the Sun at autumnal equinox.
C is the position of the Sun at summer solstice.
H is the position of the Sun at winter solstice.
A is the position of the Sun, called *aphelion*, when it is farthest from the Earth.
A^* is the projection of A against the ecliptic.

Finally, K and F, and B and I, are the points of intersection of the indicated vertical and horizontal segments with the circular path of the Sun, and L is the intersection of the indicated segments.

Denote the angle between the vernal equinox and aphelion positions, i.e., the angle JEA, by λ_A. Let r

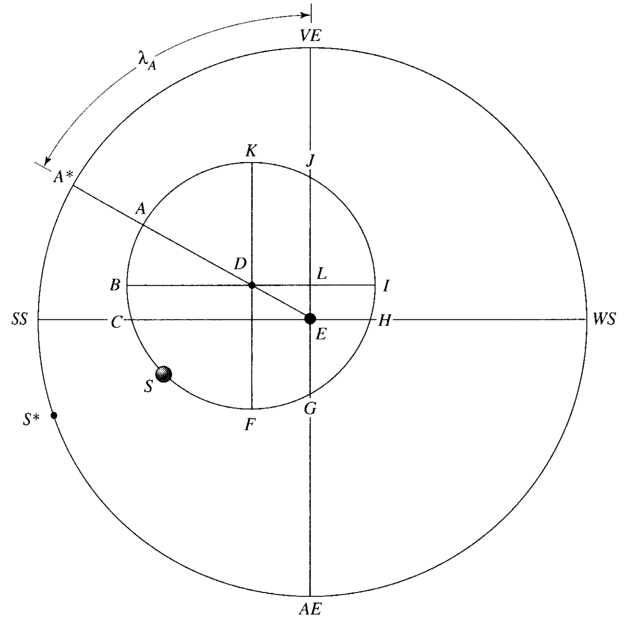


Figure 2.12

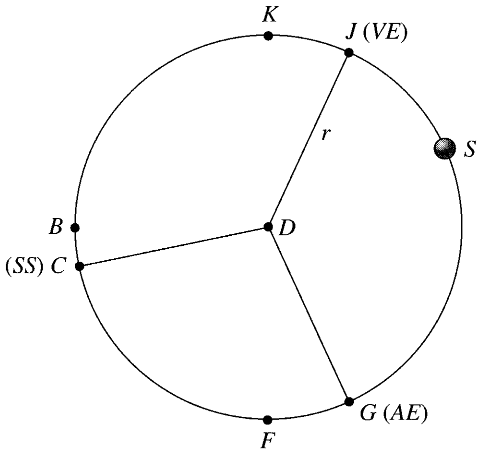


Figure 2.13

be the radius of the Sun's orbit, i.e., $r = DS$, and let e be the length of the segment ED. The solution of the problem of the precise placement of D relative to E is this: Use the information about the lengths of the seasons given by Hipparchus and determine the parameters λ_A, r, and e, or at least the ratio $\frac{e}{r}$, in such a way that the resulting scheme is consistent with these observations. The steps in Ptolemy's analysis are these: Compute the lengths of the arcs JK and BC. This gives information about the triangle DLE. This triangle then determines the parameters. The details follow. When arc JK, arc BC, etc. appear in a mathematical expression, the reference will always be to the length of the particular arc.

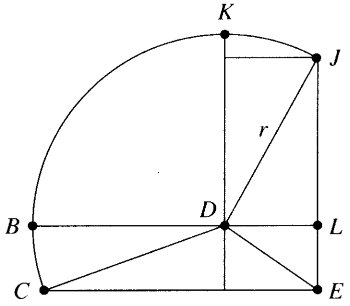


Figure 2.14

Consider the motion of the Sun S on its circular path, and let ω be the angle traced out by the rotating segment DS in one day. Since the Sun completes the full circle in $365\frac{1}{4}$ days, it follows that $\omega = \frac{360^\circ}{365\frac{1}{4}} = 0.986^\circ$. Therefore,

$$\omega = (0.986^\circ)\frac{\pi}{180^\circ} = 0.0172 \text{ radians.}$$

Now refer to Figure 2.13. Because the Sun moves from J to C during spring, which is $94\frac{1}{2}$ days long, it follows that $\angle JDC = 94\frac{1}{2}\omega = 1.625$ radians.[1] Since the radian measure of this angle is also $\frac{\text{arc}\,JC}{r}$, it follows that $\text{arc}\,JC = 1.625r$. Because summer is $92\frac{1}{2}$ days long, the very same argument shows that $\text{arc}\,CG = (92\frac{1}{2}\omega)r = 1.591r$. Therefore,

$$\text{arc}\,JC + \text{arc}\,CG = 1.625r + 1.591r = 3.216r.$$

But also, referring to Figure 2.13 again,

$$\begin{aligned}\text{arc}\,JC + \text{arc}\,CG &= \text{arc}\,JK + \text{arc}\,KF + \text{arc}\,FG\\ &= \text{arc}\,JK + \pi r + \text{arc}\,FG.\end{aligned}$$

Because arc JK = arc FG (see Figure 2.12), we now get

$$2\,\mathrm{arc}\,JK + \pi r = \mathrm{arc}\,JC + \mathrm{arc}\,CG = 3.216r.$$

Therefore, $2\,\mathrm{arc}\,JK = 3.216r - 3.142r = 0.074r$. So $\mathrm{arc}\,JK = 0.037r$. Observe, therefore, that $\angle KDJ = \frac{\mathrm{arc}\,JK}{r} = 0.037$ radians. In degrees this is approximately 2°, which is a small angle.

A look at Figure 2.14 (it has been extracted from Figure 2.12) shows that $DL \approx \mathrm{arc}\,JK = 0.037r$. More precisely, since $\theta = \angle KDJ = 0.037$ is small, $\frac{DL}{r} = \sin\theta \approx \theta = \frac{\mathrm{arc}\,JK}{r}$. (See Table 1.3 in Section 1.4.) We will therefore take

$$DL = 0.037r.$$

Ptolemy gave a similar approximation for EL. Refer back to Figure 2.13 and observe that

$$\begin{aligned}\mathrm{arc}\,BC &= \mathrm{arc}\,JC - \mathrm{arc}\,JB\\ &= \mathrm{arc}\,JC - (\mathrm{arc}\,JK + \mathrm{arc}\,KB)\\ &= 1.625r - 0.037r - \frac{\pi}{2}r\\ &= 1.625r - 0.037r - 1.571r\\ &= 0.017r.\end{aligned}$$

The angle $\angle BDC = \frac{\mathrm{arc}\,BC}{r} = 0.017$ radians is a small angle; it is approximately 1°. So by another look at Figure 2.14, $EL \approx \mathrm{arc}\,BC = 0.017r$. We will therefore take

$$EL = 0.017r.$$

Figure 2.15 shows the right triangle DLE as extracted from Figure 2.12 with the insertion of the information just derived. Ptolemy could now compute the basic parameters e and λ_A. By the Pythagorean theorem,

$$\begin{aligned}e &= \sqrt{DL^2 + EL^2} = \sqrt{(3.7 \times 10^{-2}r)^2 + (1.7 \times 10^{-2}r)^2}\\ &= \sqrt{13.69(10^{-2}r)^2 + 2.89(10^{-2}r)^2}\\ &= 10^{-2}r\sqrt{13.69 + 2.89}\\ &= 10^{-2}r\sqrt{16.58} = 0.041r.\end{aligned}$$

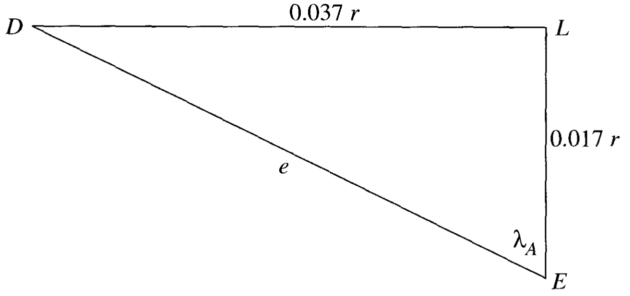


Figure 2.15

It follows that e is approximated by

$$e = \frac{1}{24}r.$$

Because $\sin\lambda_A = \frac{DL}{e} = \frac{0.037r}{0.041r} = \frac{0.037}{0.041} = 0.90$, and Ptolemy knew trigonometry, he concluded that

$$\lambda_A = 64°.$$

Ptolemy's determination of the position of the center of the Sun's circular orbit relative to the Earth was complete. The *Almagest* also contains an estimate for r (the distance from the Earth to the Sun), but it is no more accurate than that of Aristarchus. See the values for D_S in Table 1.4 of Section 1.5.

Once Ptolemy established his theory of the Sun's motion, he used it to confirm Hipparchus's observation that winter has $90\frac{1}{8}$ days. By Figure 2.12,

$$\begin{aligned}\mathrm{arc}\,HJ &= \mathrm{arc}\,HI + \mathrm{arc}\,IJ\\ &= \mathrm{arc}\,BC + (\mathrm{arc}\,IK - \mathrm{arc}\,JK)\\ &= 0.017r + \left(\frac{\pi}{2}r - 0.037r\right)\\ &= 0.017r + (1.571r - 0.037r)\\ &= 1.551r.\end{aligned}$$

Note that the angle that the Sun sweeps out during winter is $\angle HDJ$. Its radian measure is $\frac{\mathrm{arc}\,HJ}{r} = 1.551$. Since it sweeps out $\omega = 0.0172$ radians per day, it follows that winter must be $\frac{1.551}{0.0172} = 90.17 \approx 90\frac{1}{6}$

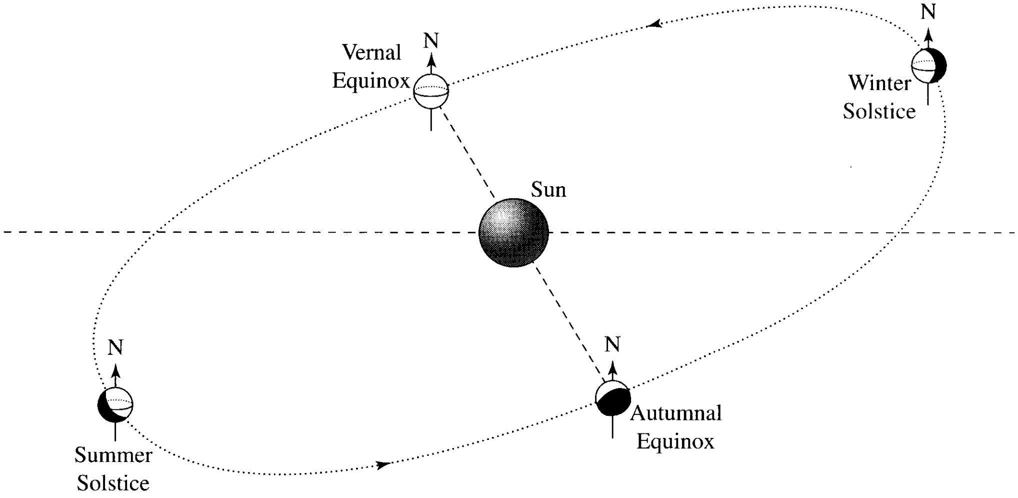


Figure 2.16

days long. This corresponds closely to the $90\frac{1}{8}$ days observed by Hipparchus.

2.4 The Modern Perspective

Let us now turn to today's heliocentric point of view. The motion of the Earth has two primary features. One is the circular orbit of the Earth with the Sun in the interior of the circle. (This orbit is in fact not quite circular. It is a circle that is "stretched" very slightly along a diameter. This subtlety will be taken up in Chapter 4.) The other is the Earth's daily rotation about the axis through the two poles. This north-south axis is perpendicular to the equator, but it is not perpendicular to the Earth's orbit. This is because the plane determined by the Earth's equator is not the same as that of the orbit. For purposes of the present discussion, think of what is going on in the following way. Consider the plane parallel to the Earth's equator and through the center of the Sun. We will call it the "plane of the Sun." See Figure 2.16. This figure show a perspective of the Sun from "above." The two dotted lines in the figure are perpendicular lines in this plane. The arrow N (for north) lies on the axis of the Earth's rotation and is, therefore, perpendicular to the plane of the Sun. This arrow always points in the same direction: north. The Earth passes alternately above and below the plane of the Sun. At summer solstice, the Sun shines down most directly on the Earth's northern hemisphere. From the perspective of Figure 2.16, the Earth is now at its lowest point below the plane of the Sun. On this day the Sun is above the horizon for the longest period of time in the northern hemisphere; it is the day of longest daylight. On the day of winter solstice, the smallest portion of the northern hemisphere is exposed to the Sun; it is the day on which the Sun is above the horizon for the shortest period of time. There are two instances where the center of the Earth lies in the plane of the Sun. Since the axis of rotation is perpendicular to this plane, it follows that on the two days when this occurs, every location on the Earth receives 12 hours of sunshine. These are the days of vernal and autumnal equinox. From the perspective of Figure 2.16, the Earth is above the plane of the Sun during its motion from autumnal equinox to winter solstice to vernal equinox (i.e., during autumn and winter); and the Earth is below the plane of the Sun as it moves from vernal equinox to summer solstice and back to

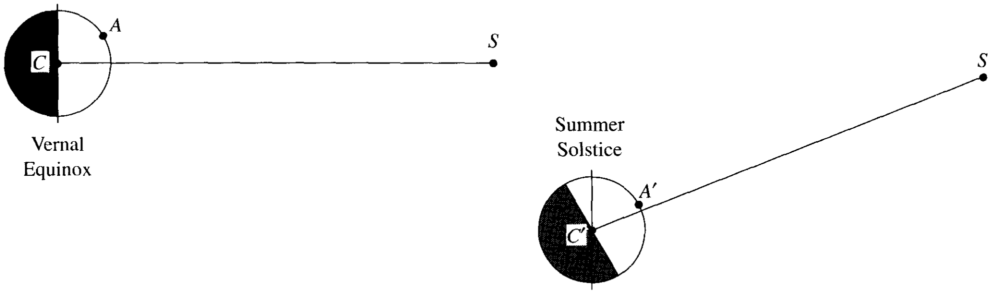
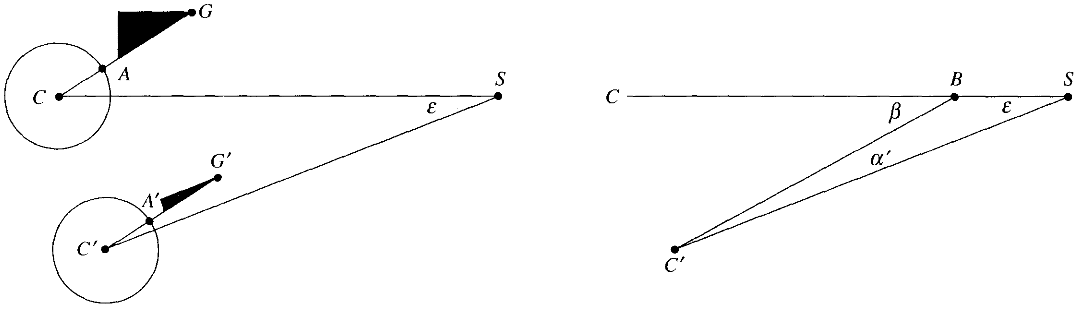


Figure 2.17



(a) **(b)**

Figure 2.18

autumnal equinox (i.e., during spring and summer). The figure indicates the dark and sunlit regions of the Earth at each of the four relevant positions.

The question that we will turn to in this section as well as the next is this: What information do the methods of the Greeks provide when they are applied to the heliocentric point of view?

Consider the plane perpendicular to the plane of the Sun and through both the center of the Sun and the center of the Earth at summer solstice. (Use Figure 2.16 to visualize this plane.) Now rotate the vernal equinox position of the Earth into this plane as shown in Figure 2.17. The Earth's axis of rotation and the regions of sunlight and shadow on the Earth are indicated in the figure. In each case, it is noon in Alexandria. The points C and C' are the center of the Earth, A and A' refer to Alexandria, and S represents the center of the Sun. Now put the diagrams of Figure 2.17 together. This is done in Figure 2.18(a). The segments AG and $A'G'$ are gnomons planted in Alexandria at noon. (They are drawn larger than to scale in order to make them and the shadows that they cast visible in the diagram.) A moment's thought shows that ε is the angle between the plane of the Sun and the plane of the Earth's orbit. Denote the angle at G by α and that at G' by α'. Notice that the angle at C' is also α'.

Figure 2.18(b) is derived from Figure 2.18(a) as follows. Extend the segment $C'A'G'$ to a point B on CS and let β be the angle CBC'. Because the segments $C'B$ and CG are both determined by a gnomon placed in Alexandria, they are parallel. Therefore, $\beta = \alpha$. The angles ε, α', and $\pi - \beta$ belong to the same triangle. So

$$\pi = \varepsilon + \alpha' + (\pi - \beta) = \varepsilon + \alpha' + (\pi - \alpha)$$

and hence $\varepsilon + \alpha' - \alpha = 0$. It follows that $\varepsilon = \alpha - \alpha'$.

This discussion has put one of the shadow measurements of the Greeks (as discussed in Section 2.1) into the heliocentric context. By measuring the angles α and α' and computing the obliquity of the ecliptic $\varepsilon = \alpha - \alpha'$, the Greeks had in effect computed the angle between the plane of the Earth's equator and the plane of the Earth's orbit about the Sun. Recall that the Greeks measured ε to be 24°. A more accurate value for this angle is $23\frac{1}{2}^\circ$.

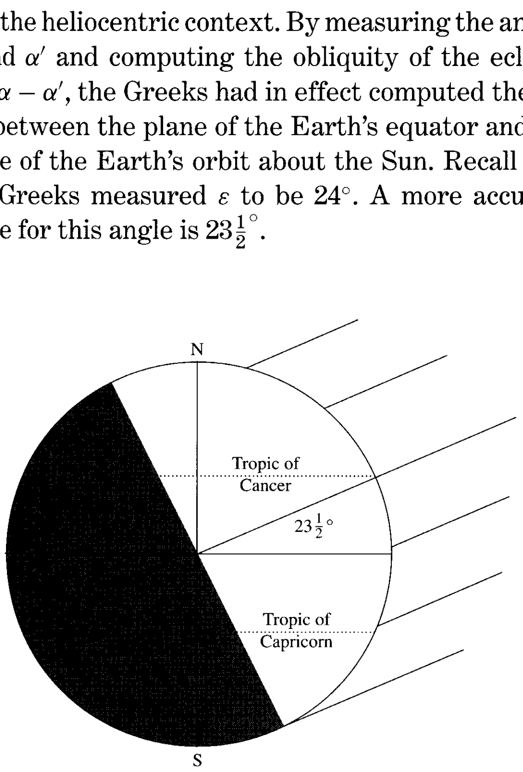


Figure 2.19

Figure 2.19 is a "close-up" of the summer solstice situation in Figure 2.17. A look at the geometry shows that the Sun will be directly overhead at noon at any point on the surface of the Earth that lies $23\frac{1}{2}^\circ$ above the equator. The line that this angle determines is the tropic of Cancer. The tropic of Capricorn is the corresponding line for the Southern Hemisphere.

Consult an atlas and confirm that Aswan, Egypt, lies very near the tropic of Cancer. We now understand why the Sun at summer solstice shines into the very bottom of the well in Syene (the ancient city near today's Aswan). Recall that this fact was instrumental in Eratosthenes's computation of the radius of the Earth.

2.5 Another Look at the Solar Model

This section turns to the questions: What quantitative information do Ptolemy's mathematical methods provide about the circular orbit of the Earth around the Sun? What, in particular, do they tell us about the position of the Sun within the Earth's orbit?

We will make use of today's values for the lengths of the seasons. Spring: 92 days, 18 hours, 20 minutes, or 92.764 days; summer: 93 days, 15 hours, 31 minutes, or 93.647 days; autumn: 89 days, 20 hours, 4 minutes, or 89.836 days; and winter: 88 days, 23 hours, 56 minutes, or 88.997 days.[2] This is the information that will be fed into Ptolemy's model of Section 2.3.

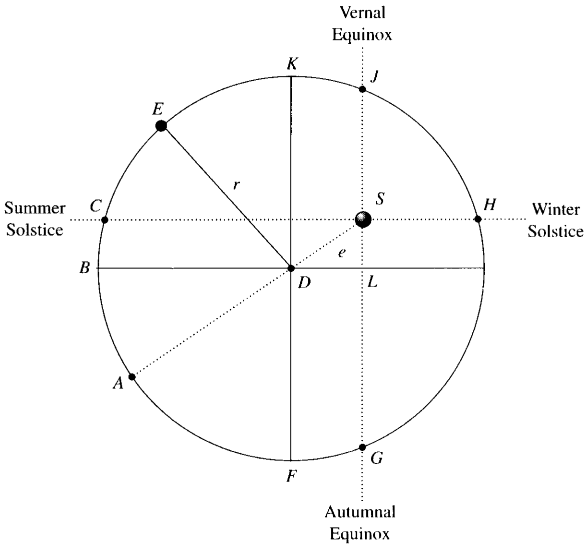


Figure 2.20

Figure 2.20 shows the Earth's circular orbit. The points S and E represent the Sun and the Earth, respectively. The point D is the center of the Earth's orbit. The radius of the circle is r, and e is the distance from S to D. The points J, C, G, and H are the positions—now of the Earth—at vernal equinox, summer solstice, autumnal equinox, and winter solstice. Since summer is the longest season, the center D of the circle must be positioned below and to the left in relation to the Sun. Why? Aphelion, the position of the Earth farthest from the Sun, is labeled A. We will suppose that the Earth moves with a constant angular velocity of $\omega = \frac{360^\circ}{365\frac{1}{4}} = 0.986^\circ/\text{day} = 0.0172$ radians/day.

During spring, the Earth moves from J to C. Looking at the radian measure of $\angle JDC$ in two ways, we get

$$\angle JDC = \frac{\operatorname{arc} JC}{r} = 92.764\omega$$
$$= (92.764)(0.0172) = 1.596.$$

So $\operatorname{arc} JC = 1.596r$. Doing the same for summer shows that

$$\angle CDG = \frac{\operatorname{arc} CG}{r} = 93.647\omega$$
$$= (93.647)(0.0172) = 1.611.$$

So $\operatorname{arc} CG = 1.611r$, and $\operatorname{arc} JCG = 1.596r + 1.611r = 3.207r$. On the other hand, since $\operatorname{arc} JK = \operatorname{arc} FG$, we have $\operatorname{arc} JCG = 2 \operatorname{arc} JK + \pi r$. Therefore, $2 \operatorname{arc} JK = 3.207r - 3.142r = 0.065r$. It follows that

$$DL \approx \operatorname{arc} JK = 0.033r.$$

As before, we will regard this and subsequent approximations as equalities. Since

$$\operatorname{arc} CG = \operatorname{arc} CB + \frac{\pi}{2} r + \operatorname{arc} FG$$
$$= \operatorname{arc} CB + 1.571r + \operatorname{arc} JK,$$

we get

$$1.611r = \operatorname{arc} CB + 1.571r + 0.033r,$$

or $\operatorname{arc} CB = 1.611r - 1.571r - 0.033r = 0.007r$. So

$$SL \approx \operatorname{arc} CB = 0.007r.$$

Now take $DL = 0.033r$ and $SL = 0.007r$ and consider the right triangle SDL in Figure 2.21. By Pythagoras,

$$e = \sqrt{DL^2 + SL^2} = \sqrt{(33 \times 10^{-3} r)^2 + (7 \times 10^{-3} r)^2}$$
$$= \sqrt{1089(10^{-3}r)^2 + 49(10^{-3}r)^2} = 10^{-3} r \sqrt{1089 + 49}$$
$$= 10^{-3} r \sqrt{1138} = 0.034r.$$

Since $\sin\theta = \frac{DL}{e} = \frac{0.033r}{0.034r} = \frac{0.033}{0.034} = 0.970$, we can conclude (use the inverse sine button of a calculator) that $\theta = 76^\circ$.

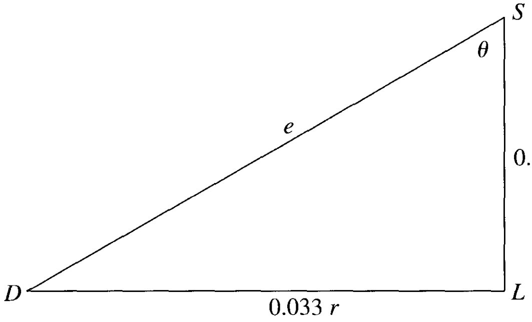


Figure 2.21

Ptolemy's mathematics has produced a model of the Earth's orbit around the Sun. This orbit is a circle that has the Sun at a distance e from the center. While we know today that the Earth's orbit is in fact an ellipse, we will see in Chapter 4 that this ellipse is very nearly a circle. We also know that the Sun is not at the center of the ellipse, but that it is off-center (at a focus). Ptolemy's model, in other words, agrees with the essence of the actual situation. We will now consider other elements that Ptolemy's model captures.

The distance from the Earth to the Sun is known to vary by about 1.7%. This means that one-half the difference between the maximum and minimum distances divided by the average distance is about 0.017. Let's now return to the model. Here, too, the distance from the Earth to the Sun varies. Refer back to Figure 2.20 and note that the maximum distance (attained in the summer) is $AD + DS = r + e$, and the minimum distance (attained in winter) is $r - e$. So one-half the

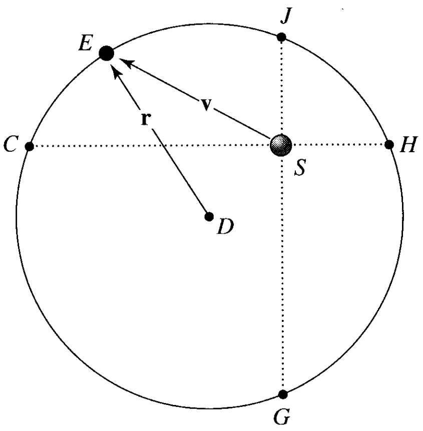


Figure 2.22

difference between the maximum and minimum distance is $\frac{1}{2}[(r+e)-(r-e)] = \frac{1}{2}(2e) = e$, and the average distance is $\frac{1}{2}[(r+e)+(r-e)] = r$. Therefore, the ratio $\frac{1}{2}$(max − min)/average is

$$\frac{e}{r} = 0.034,$$

or about 3.4%. This ratio is twice the actual value. Again, however, the model reflects the phenomenon.

Figure 2.22 shows the Earth E in its circular orbit around the Sun S with D the center of the orbit. It is extracted from Figure 2.20. Let $\mathbf{r}$ be the arrow from D to E and recall that $\mathbf{r}$ rotates with constant angular velocity. Let $\mathbf{v}$ be the arrow from S to E. During winter, E moves from H to J (winter solstice to vernal equinox). In the process, $\mathbf{v}$ rotates from H to J and hence through exactly 90°. At the same time, $\mathbf{r}$ also rotates from H to J, and hence through an angle less than 90°. Therefore, during winter $\mathbf{v}$ rotates, on average, faster than $\mathbf{r}$. During summer, both $\mathbf{r}$ and $\mathbf{v}$ rotate from C to G. Again $\mathbf{v}$ rotates through 90°, but $\mathbf{r}$ now rotates through more than 90°. So during summer $\mathbf{v}$ rotates more slowly than $\mathbf{r}$. Since $\mathbf{r}$ rotates at constant angular velocity, it follows that $\mathbf{v}$ has, on average, greater angular velocity during winter, when the Earth is closer to the Sun, than during summer, when it is farther away. This is in fact true, and consistent (as we shall see in Chapter 4) with Kepler's second law of planetary motion!

2.6 Epicycles

Let's return to the *Almagest*. Having dealt with the Sun, Ptolemy next turned to the Moon. The theory of the Moon is the most challenging. Indeed, the *Almagest* presents a progression of three different lunar theories of increasing complexity, a complexity forced by persistent discrepancies in lunar observations. In retrospect, this is hardly surprsing. The distance from the Earth to the Moon varies by more than 6% over a *month* (the period of its orbit). The Moon can get as close as 220,000 miles to the Earth and as far as 250,000 miles away. While the orbit is in the general shape of an ellipse, it is not a closed loop. This complexity is explained by the fact that the Moon is subject to the gravitational forces of both the Earth and the Sun. The mathematics of the interplay of the gravitational forces of three masses is known as the three-body problem. It is very difficult and still not completely solved today.

Another phenomenon that Ptolemy's astronomy had to explain was the *retrograde* motion of the planets. The planets—as observed from the Earth against the background of the stars—move near the ecliptic. However, their motion is irregular. They are observed to intermittently slow down, stop, reverse direction for some time (this is the *retrograde* phase), only to stop once more and then continue in the original direction. For example, the apparent path of Saturn as Ptolemy observed it in the years 132–133 A.D. is pictured in Figure 2.23.

Given the Sun-centered perspective, the explanation of what Ptolemy observed is straightforward. The Earth completes one orbit around the Sun in a year, while Saturn takes about 30 Earth years to complete an orbit. Figure 2.24 shows the (dotted) lines of sight of someone on Earth looking out on Saturn against the background of the fixed stars, just as Ptolemy had done. Observe that the general pattern matches

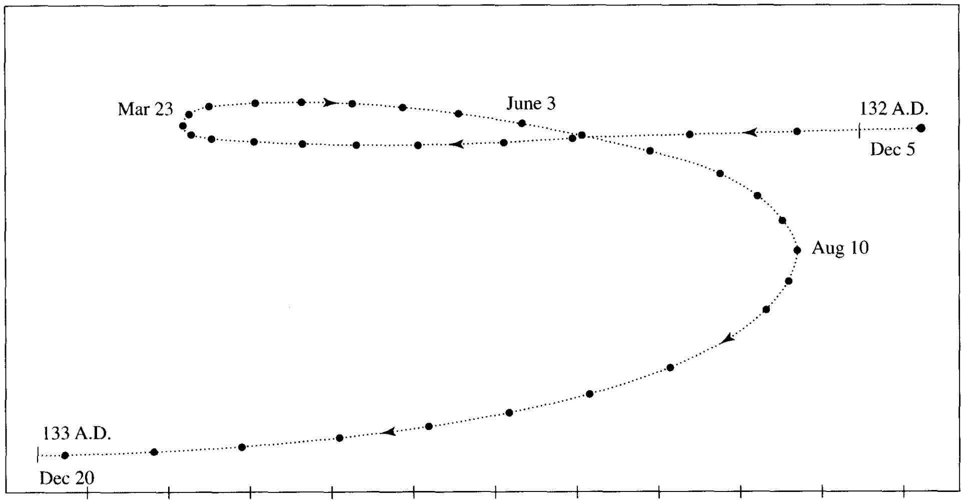


Figure 2.23

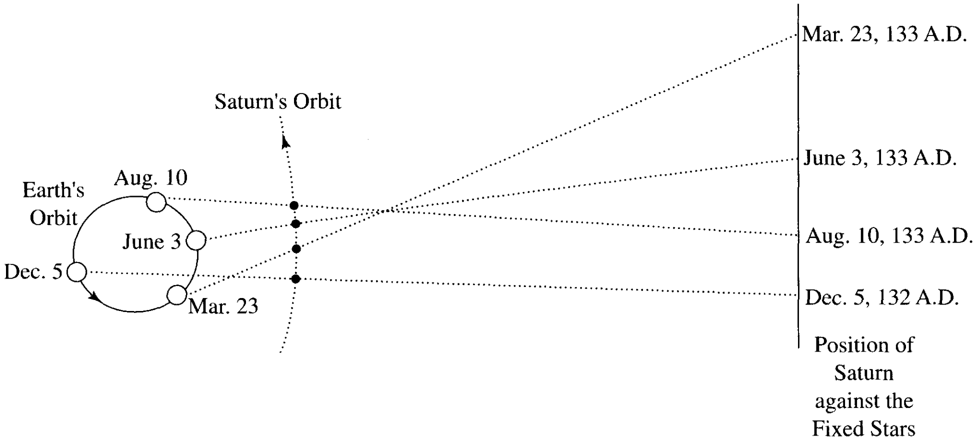


Figure 2.24

that of Figure 2.23. The observation that the planets move near the ecliptic is explained by the fact that the planes of their orbits are almost the same as that of the Earth's orbit. The planes are not exactly the same, however, and this explains the fact that the planets are observed to rise and fall. For instance, Figure 2.23 shows that Saturn is higher in the sky in December of 132 A.D. than in December of 133 A.D.

How does the *Almagest* deal with the irregularities of the Moon's orbit, and how does it—with its Earth-centered perspective—come to grips with the phenomenon of the retrograde motion of the planets? The answer, in short, is that it combines circular motions in a variety of ways. Let's focus on the retrograde motion of the planets. Figure 2.25 shows a planet P in a circular orbit with radius r and center the point B. In turn, B moves along another circle. It has the

Earth E as its center and radius R. The larger circle is called the *deferent* and the smaller one the *epicycle*. Both rotations occur at constant speeds. The arrows designate the directions. Suppose that P moves faster around B than B does around E. When P is in the position shown, the two motions—as viewed from E—are in opposite directions. But P will be seen to move from left to right due to the greater speed of P. However, as P continues around B, it will eventually move in the same direction as B. Now P will be seen to be moving from right to left. So this scheme of two rotating circles explains, at least in principle, the observed reversal of a planet's direction. This model is the basis of the *Almagest*'s lunar and planetary theories.

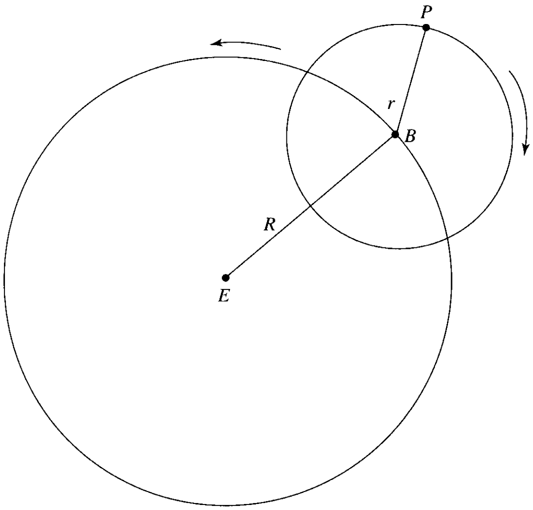


Figure 2.25

To make his theory stay in tune with his observations, Ptolemy was forced to make adjustments to the scheme of Figure 2.25. These are indicated in Figure 2.26. He modified the orbit of the planet P by moving the center of the deferent to a point C away from E. In addition, he corrected the speed of the point B by choosing yet another point A and stipulating that the segment from A to B rotates with constant speed. The various parameters—the positions of C and A relative to E, the radii R and r, and the speeds of the rotations—were different for each planet. When even this scheme did not mesh with observations, he introduced additional circles. Different configurations of circles were required. Those for the inner planets (Mercury and Venus) were more complicated than those of the outer planets (Mars, Jupiter, and Saturn), and (as already remarked), the Moon's was the most complicated of all. In summary, Ptolemy's solar system was an extremely intricate clockwork of spokes and wheels.

2.7 Postscript

Our overview of the *Almagest* is complete. Given the parameters it starts with—an Earth-centered solar system in which celestial objects move in circular orbits at constant speeds—it succeeded in a remarkable way. We have seen how it combines mathematics and empirical data derived from careful observations to produce a mathematical model that simulates the Sun's orbit about the Earth. It is thus an early example of mathematical physics.

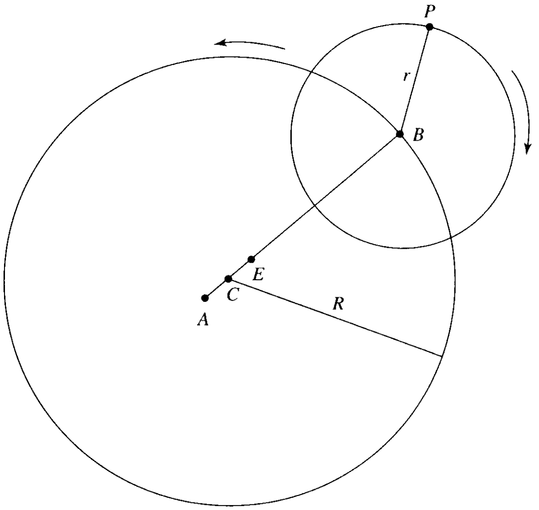


Figure 2.26

The *Almagest* is unrivaled in antiquity as an example of how a large and important class of natural phenomena could be described in mathematical terms. On the whole, it is to be viewed not as a description of the physical universe but rather as an abstract mathematical scheme designed to predict the positions of the objects moving within it. In this sense it agreed sufficiently with observations to be acceptable to practical astronomers. Observe that from the perspective of a fixed Earth, the planets do (in essence) move along epicycles carried by the deferent of the Sun's orbit about the Earth (see Figure 2.26). Therefore, from the perspective of the Earth, Ptolemy's is the correct geometry.

The *Almagest* contains a catalogue of stars (1022 of them). This was necessary because crucial observations (those of the planets for instance) are made in reference to them. Arab astronomers expanded this catalogue and compiled atlases of stars that proved invaluable to early Spanish and Portugese navigators. Ptolemy's *Almagest* remained the foundation of astronomy for fourteen centuries. In fact, the scheme of Figure 2.1 is still the most common coordinate system for listing planetary and stellar positions. An observer on the Earth can specify the location of a star X by measuring α (called the star's *right ascension*) and δ (called its *declination*) in terms of angles from the point of the observation; see Figure 2.27.

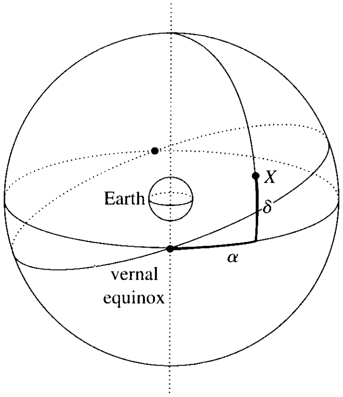


Figure 2.27

Ptolemy's clockwork did not become obsolete until the 16th and early 17th centuries, when Copernicus introduced the heliocentric universe (reintroduced is more appropriate, in view of Aristarchus), Galileo discovered the phases of Venus and the moons of Jupiter, and Kepler demonstrated that the orbits of the planets were ellipses.

Exercises
2A. Some Geometry

1. Consider any rectangle and let A, B, C, and D be its vertices. Inscribe it into a circle with center at the intersection of the diagonals. Use Ptolemy's theorem to verify Pythagoras's theorem.
2. Place an angle θ at the center of a circle of radius 2. Let the endpoints of the arc that it cuts from the circle be A and B. What does θ equal (in degrees) if the length of the arc from A to B is $1\frac{1}{2}$?
3. Refer to Figure 2.28. The points A, B, and D are on a circle of radius 3. The length of the arc AB is 4. Determine the angle $\angle ADB$ first in radians and then in degrees.

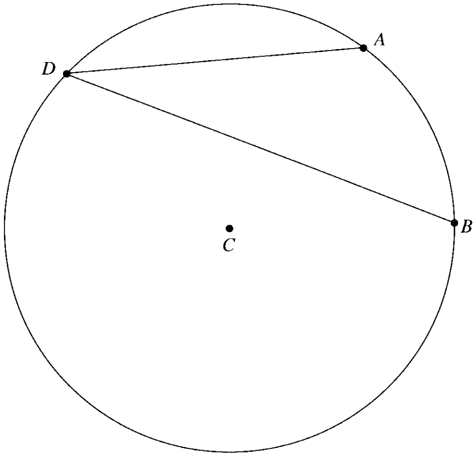


Figure 2.28

2B. Some "Inverse" Trigonometry

4. Use a calculator to fill in the following blanks (use three decimal accuracy):

i. $\sin$ ____$^\circ = 0.534$
ii. $\sin$ ____$^\circ = 0.219$
iii. $\sin$ ____$^\circ = 0.834$
iv. $\sin$ ____ (in radians) $= 0.002$
v. $\sin$ ____ (in radians) $= 0.664$
vi. $\tan$ ____$^\circ = 0.774$
vii. $\tan$ ____$^\circ = 1.478$
viii. $\tan$ ____$^\circ = 5.000$
ix. $\tan$ ____ (in radians) $= 10.473$
x. $\tan$ ____ (in radians) $= 27.664$

5. What angle α (in degrees and radians) has $\sin\alpha = \frac{1}{2}$? What angle β (degrees and radians) has $\tan\beta = 1$? What angle φ has $\sin\varphi = \frac{\sqrt{3}}{2}$?

2C. More Trigonometry

6. Let α and β be two angles, both less than $\frac{\pi}{2}$, and arrange them into a triangle as shown in Figure 2.29.

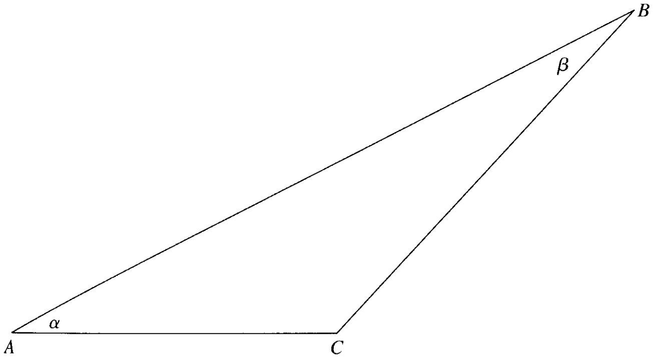


Figure 2.29

Complete this triangle to Figure 2.30 in such a way that the segment AD is perpendicular to BD; also draw the perpendicular EC to AB.

i. Show that $\gamma = \alpha + \beta$.
ii. Show that $\frac{BD}{AB} = \frac{EC}{AC}$ and hence that $BD = \frac{EC}{AC}(AE + EB)$.
iii. Using (i) and (ii), show that $\sin(\alpha + \beta) = (\sin\alpha)(\cos\beta) + (\cos\alpha)(\sin\beta)$.

It can also be shown (we will forego the pleasure) that

iv. $\cos(\alpha + \beta) = (\cos\alpha)(\cos\beta) - (\sin\alpha)(\sin\beta)$.

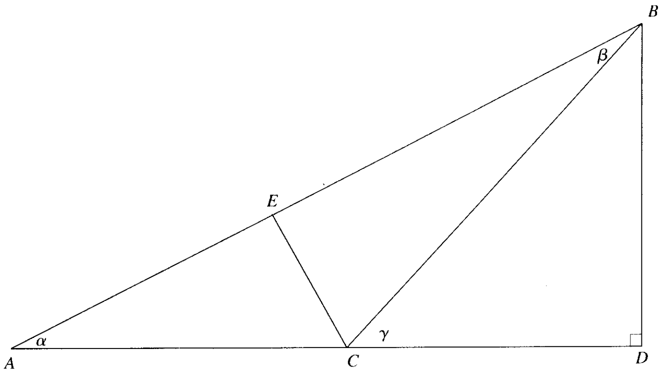


Figure 2.30

Note: The identities (iii) and (iv) are crucial for the development of the calculus of the trigonometric functions.

7. Let θ be any angle, and place it into a circle of radius 1, center C, and diameter DA, as shown in Figure 2.31. Draw the segments BD and BC, and draw a perpendicular BE down to DA.

i. Show that $\angle BDC = \frac{\theta}{2}$.
ii. Show that $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$.
iii. Deduce from (ii) that $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$. [Hint: Multiply (ii) by $\frac{1-\cos\theta}{1-\cos\theta}$.]
iv. Use (ii) and (iii) to show that $1 - \tan^2\frac{\theta}{2} = \frac{2\cos\theta}{1+\cos\theta}$.
v. Use (ii) and (iv) to show that $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$.

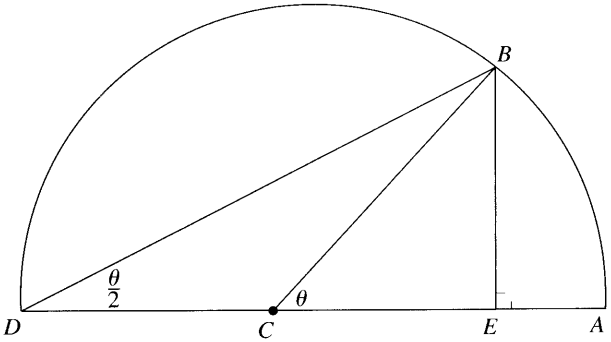


Figure 2.31

2D. Ptolemy's Mathematics

8. Redo the calculations of Section 2.3 by starting with $\omega = 0.02$ and by working to an accuracy of two decimal places. What values for the ratio $\frac{e}{r}$ and the angle λ_A do you get? What does this tell us about Ptolemy's calculations?

9. Use the model of the Earth's orbit around the Sun in Section 2.5 to determine the days of the year on which the Earth is farthest (*aphelion*) and closest (*perihelion*) to the Sun. Compare this with the data given in the *World Almanac*. [Hint: Summer solstice occurs on June 21 or June 22, and the Earth is in position C in Figure 2.20 on that day. How long will it take for the Earth to get to position A?]

10. Ptolemy computed the distance D_M between the Earth and the Moon as follows. In Figure 2.32, E and M are the centers of the Earth and Moon, respectively, and r_E is the radius of the Earth. Take $r_E = 3,850$ miles from Eratosthenes. The point A represents Alexandria and AQ is perpendicular to EM. Ptolemy observes the angle α to be $50°55'$ and computes β (from lunar longitude and obliquity of ecliptic data) to be $49°48'$. Supply the details to the following outline of Ptolemy's argument:

  i. Show that $\alpha = \beta + \mu$.
  ii. Show that $D_M = r_E \cdot \frac{\sin\alpha}{\sin\mu}$ using the addition formula for the sine.
  iii. Obtain $D_M = 150{,}000$ miles.
  iv. Compare this with the modern value.

2E. Diophantus of Alexandria

Diophantus (of Alexandria, about 250 A.D.) was a mathematician who pursued arithmetic and algebra.

11. The following passage from *The Greek Anthology*[3] provides some information about Diophantus's life. Use it to determine his age. Diophantus passed one-sixth of his life in childhood, one-twelfth in youth, and one-seventh more as a bachelor. Five years after his marriage was born a son who died four years before his father, at half of his father's [final] age.

12. Solve the following problem from Diophantus's book *Arithmetica*: Find four numbers, the sum of every arrangement three at a time being 22, 24, 27, and 20.

13. Solve the following, also from the *Arithmetica*: In the right triangle ABC, right angled at C, AD bisects the angle at A. Find the set of smallest integers for AB, AD, AC, BD, DC, such that the ratios DC:CA:AD are equal to 3:4:5. [Hint: Use of the formula from Exercise 7(v) and the fact that if a prime number divides a product of two integers, then it must divide one of the factors.]

Notes

[1] The computations that follow are carried out with three decimal accuracy, even though the data on which they are based (the length of the seasons and the year) do not meet this standard. These computations thus violate today's principles of reliability. However, they do reflect Ptolemy's own both in

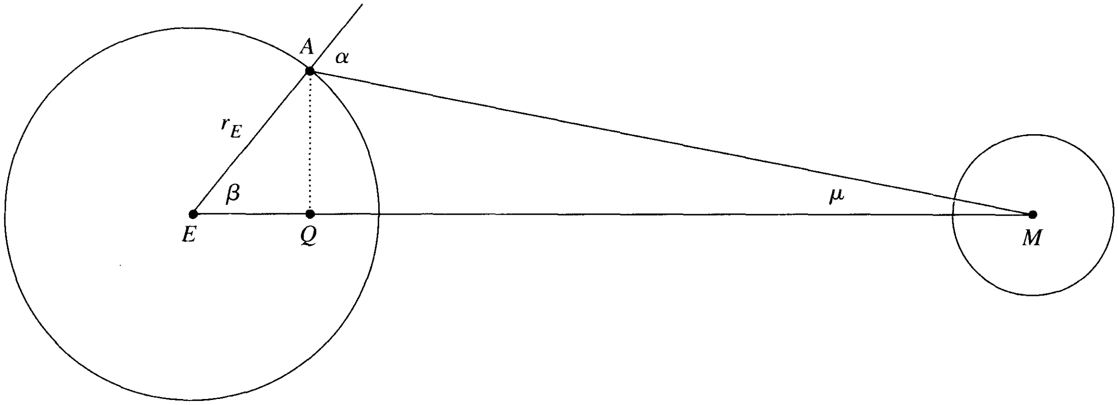


Figure 2.32

spirit and in terms of their numerical conclusions. The computations take $\pi = 3.142$.

[2] The reason for the difference between the lengths of the seasons computed from Ptolemy's time and now has to do with the fact that the year (the number of days between successive summer solstices, or vernal equinoxes) is slightly less than the $365\frac{1}{4}$ days measured by Hipparchus. It is actually 365 days, 5 hours, 48 minutes, and 46 seconds long (11 minutes and 14 seconds less than $365\frac{1}{4}$ days).

[3] See Note 8 of Chapter 1.

3

Archimedes Computes Areas

Several centuries have passed since the achievements of Claudius Ptolemy. During this time, Christianity emerged and the Germanic invasions brought instability. Faith and survival, not mathematics and science, became important. Islam conquered North Africa, the Middle East, and parts of Europe. The Latin West was thus cut off from regular contact with the Greek East, and the mathematical and scientific discoveries that we have described were largely lost in Western Europe between 650 A.D. and 1150 A.D. It was not until the beginning of the 13th century that the works of the Greek scholars were rediscovered in Arabic manuscripts and translated into Latin. This stimulated the pursuit of scientific thought (much of it dominated by attempts to understand Aristotelian science in its relationship to Christian revelation) in the new universities that had sprung up by that time.

The Renaissance—the rebirth—began in Italy in the 14th century. Greek philosophy, architecture, art, science, and mathematics provided a crucial stimulus. The recovery of Ptolemy's maps[1] reintroduced the method of defining geographical position in terms of coordinates, that is to say, by the circles of latitude and the meridians of longitude on the globe. The Western world learned the mathematics and astronomy of Euclid, Aristarchus, Archimedes, Apollonius, and Ptolemy. The interaction and confrontation with Greek science generated the energies that would later help bring about the invention of calculus and the scientific revolution.

We will focus on what the Europeans learned from the Greeks about three remarkable families of curves: the parabola, the ellipse, and the hyperbola. These *conic sections*—obtained by intersecting a cone with a plane—were well known to Euclid. They were analyzed extensively in the encyclopedic treatise *Conics* of Apollonius of Perga. The parabola and the ellipse—as we will see in subsequent chapters—form the basis for our understanding of both the motion of projectiles and the orbits of the planets around the Sun. After a look at some basic properties of these curves, we will turn to the ingenious mathematical discoveries that Archimedes made about them.

3.1 The Conic Sections

We will recall—without proofs—a few of the basic properties of the conic sections that we will need later. A unit of length is given. As before, if, say, P and Q are two points in the plane, PQ or QP will be the segment between them; if PQ appears in an equation, it will be understood to be the length of the segment PQ.

In the plane, fix both a line D and a point F not on D. The collection of all points P such that the length

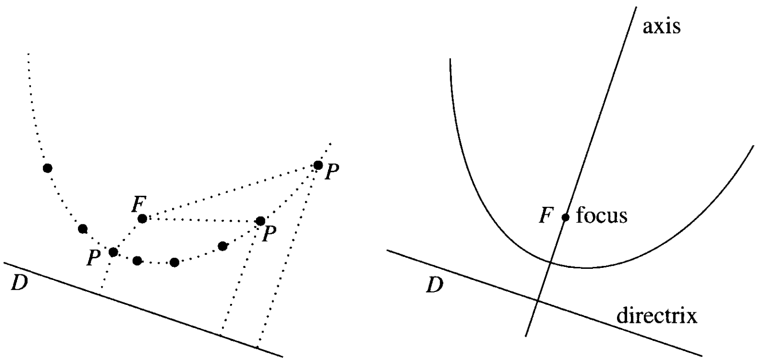


Figure 3.1

of PD is equal to that of PF is called a *parabola*. (See Figure 3.1.) The line D is called the *directrix* and the point F is called the *focus*. The line through the focus that is perpendicular to the directrix is the *axis* of the parabola.

Proposition 3.1. Let P be any point on a parabola and let the segment AB be tangent to the parabola at P. Consider the line from P that is parallel to the axis of the parabola and let Q be a point on this line, as shown in Figure 3.2. Then $\angle QPA = \angle FPB$.

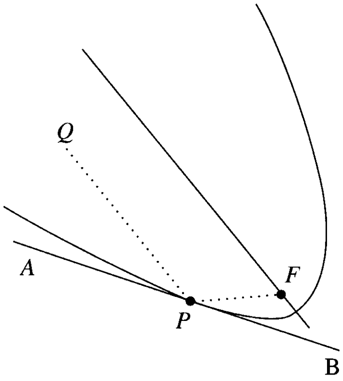


Figure 3.2

This is the "reflection property" of the parabola. We point out in passing that when light is reflected by a mirror, the angle of incidence is equal to the angle of reflection. It follows that if a mirror has the shape obtained by rotating a parabolic arc one revolution around its axis, then any light ray parallel to the axis will be reflected through the focus. This is the reason for the use of parabolic mirrors both in telescopes (for collecting light) and in the headlights of cars (for projecting light).

Again, consider any parabola. Cut it with any straight line, and let S and S' be the points of intersection. (See Figure 3.3(a).) For some point V on the parabola, the tangent line at V is parallel to the cut SS'. The parabolic region SVS' is called a *parabolic section*, and V is the *vertex* of the parabolic section. A *section*, generally speaking, is a region obtained from another by a cut. Let A be the midpoint of the segment SS'; connect A with V, and take E on the tangent such that ES is parallel to VA. (Refer to Figure 3.3(b).) Finally, let B be any point on VE and let C be the point on the parabola such that BC is parallel to VA. (See Figure 3.4.)

Proposition 3.2. Take B to be the midpoint of VE. Then C is the vertex of the parabolic section SVC. (See Figure 3.5.)

Proposition 3.3. For any point B on EV (see Figure 3.4),

$$\frac{BC}{ES} = \frac{VB^2}{VE^2}.$$

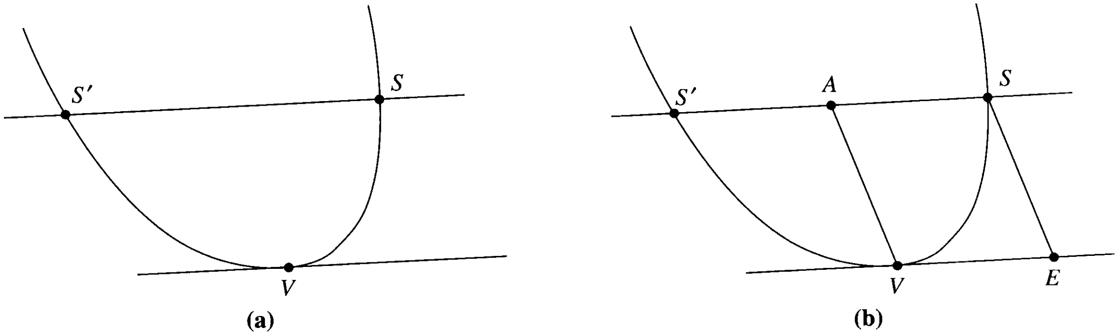


Figure 3.3

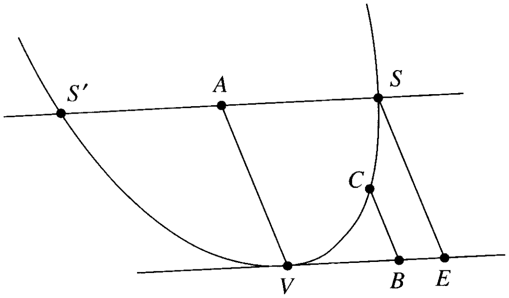


Figure 3.4

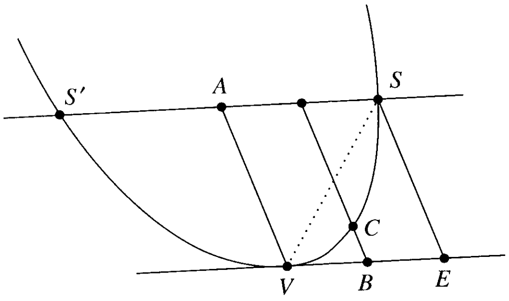


Figure 3.5

We will study the ellipse next. Fix any two points F_1 and F_2 in the plane and a constant k that is greater than or equal to the distance between F_1 and F_2. Consider the collection of all points P such that the distance from P to F_1 plus the distance from P to F_2 is equal to k. Refer to Figure 3.6(a). Any such collection of points is an *ellipse*. The points F_1 and F_2 are the *focal points* of the ellipse. The midpoint C of the segment F_1F_2 is called the *center* of the ellipse. Any segment between two points on the ellipse and through the center is a *diameter* of the ellipse. The diameter through the two focal points is the *major axis* of the ellipse, and the diameter perpendicular to the major axis is the *minor axis*. See Figure 3.6(b). Consider a circle with center C and radius r. From the preceding point of view, it is an ellipse with focal points $C = F_1 = F_2$ and $k = 2r$.

Proposition 3.4. Let P be any point on an ellipse and let the segment AB be tangent to the ellipse at P, as shown in Figure 3.7. Then $\angle F_1PA = \angle F_2PB$.

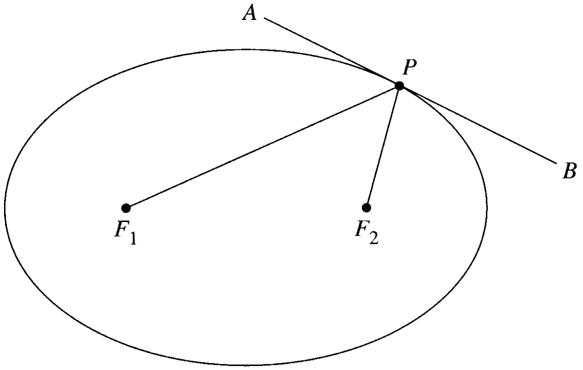
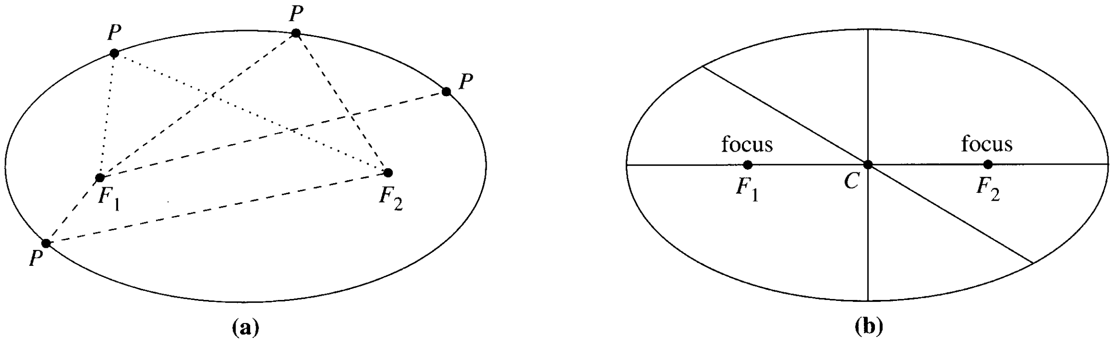


Figure 3.7



(a)

(b)

Figure 3.6

This is the "reflection property" of the ellipse. We note in passing that it is used in a surgical procedure to remove kidney stones. A source of high intensity sound waves is located at a point, say F_1, and an elliptical reflector is placed in such a way that F_1 is one of the focal points. The patient is positioned so that the kidney stone is at the other focus, F_2. The sound waves emanating at F_1 scatter in all directions, but those that strike the reflector are reflected through F_2. The strong concentration of sound waves at F_2 shatters the kidney stone.

Proposition 3.5. Let PQ be any diameter of an ellipse and let B be any point on PQ. Choose A such that AB is parallel to the tangent at P. (See Figure 3.8.) Then the ratio $\frac{QB\cdot BP}{AB^2}$ is the same no matter where B is chosen on PQ.

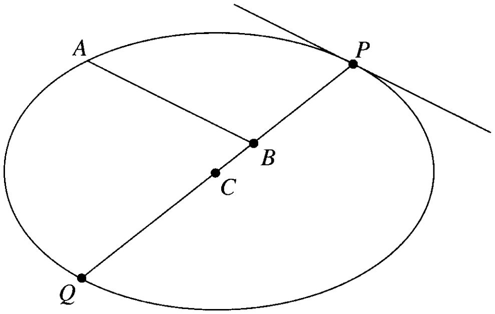


Figure 3.8

Example 3.1. Consider Proposition 3.5 in the case of a circle. See Figure 3.9. Because the focal points coincide and are located at the center C, it follows by Proposition 3.4 that the tangent at P is perpendicular to the diameter QP. Since AB is parallel to the tangent, AB and QP are perpendicular. By an application of Proposition 2.1 of Section 2.2, the triangle ΔPAQ has a right angle at A. (See Exercise 1.) Compare the triangles ΔPBA and ΔPAQ. They both have right angles and they have $\angle APB$ in common; it follows that they are similar. By the same argument, ΔQBA is also similar to ΔPAQ. Therefore, ΔPBA is similar to ΔQBA, and hence $\frac{PB}{AB} = \frac{AB}{QB}$. So, $\frac{QB\cdot PB}{AB^2} = 1$. We have shown that in the case of the circle, the ratio of Proposition 3.5 is always equal to 1.

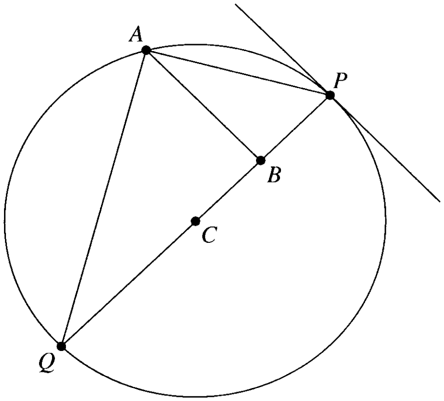


Figure 3.9

The hyperbola is defined next. Again, fix two points F_1 and F_2 in the plane and a constant $k \geq 0$. Suppose that k is less than or equal to the distance between F_1 and F_2. The collection of points P such that the absolute value of the difference between PF_1 and PF_2 is equal to k is a *hyperbola.* (Refer to Figure 3.10.) The points F_1 and F_2 are the *focal points* of the hyperbola. The line determined by the two focal points is called the *axis*. The midpoint C between the two focal points is the *center* of the hyperbola.

Hyperbolas will play only a minor role in this text. As an aside, note that space probes designed to investigate an outer planet are often redirected by allowing them to pass near an inner planet. The gravitational field of the inner planet will accelerate a probe and bend its path along a hyperbolic curve before allowing it to escape and continue.

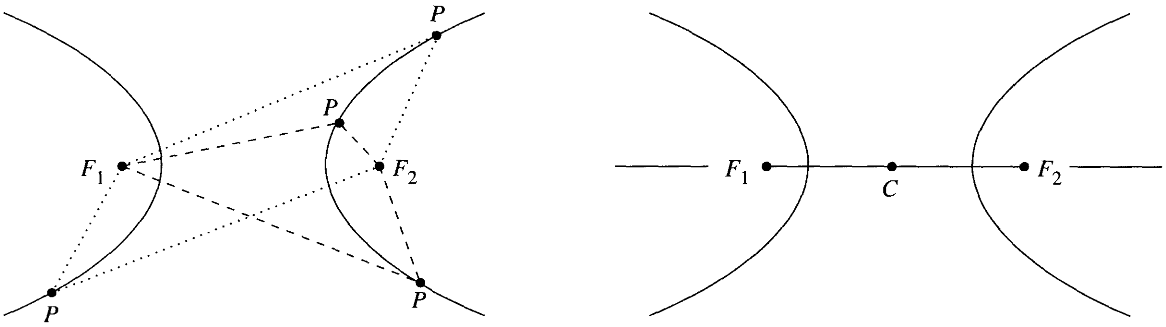


Figure 3.10

3.2 The Question of Area

We now turn to the concept of area for regions in the plane. If b is the base of a rectangle and h the height (Figure 3.11), then its area A is simply defined to be $A = bh$. This is the basic situation, and the determination of the area of any other region is ultimately reduced to this case. Consider next a parallelogram with base b and height h. By cutting off the indicated triangle on the left and reattaching it on the right, as shown in Figure 3.12, the given parallelogram is transformed—without change in area—into a rectangle with the same base b and height h. It follows that the area A of the parallelogram is $A = bh$.

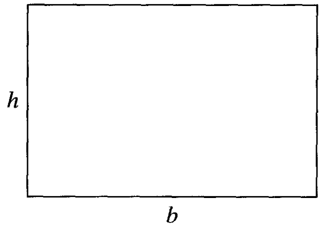


Figure 3.11

Turn next to the case of a triangle with base b and height h. Take a copy of the same triangle, flip it, and attach it to the original, as shown in Figure 3.13. The resulting larger figure is a parallelogram with

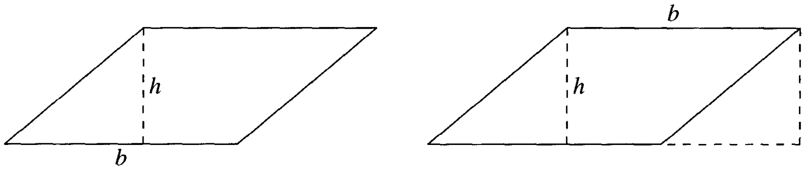


Figure 3.12

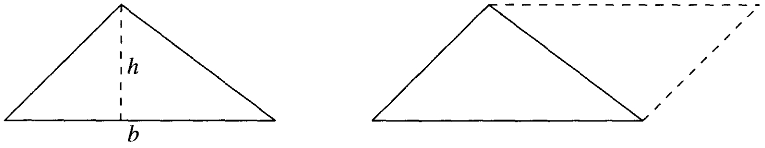


Figure 3.13

base b and height h. This larger area is equal to bh, as shown before. It follows that the area A of the original triangle is $A = \frac{1}{2}bh$.

We next consider the more subtle situation of the area of a sector of a circle with radius r. A *sector* is a pie-shaped region. Consider the sector determined by the center C and the segments CB and CD and let θ be the angle of the sector; see Figure 3.14. We will determine the area A of the sector by thinking of it as consisting of lots of very thin triangular wedges.

Partition the given sector into, say, n smaller sectors all of the same size. This determines $n - 1$ points between B and D. Consecutively connecting all of these points inscribes an isosceles triangle in each of the sectors. All of these triangles have the same shape and size. The base as well as the height of the isosceles triangles certainly depend on n, so we denote them by b_n and h_n respectively. The case $n = 5$ is shown in Figure 3.15. The bases of the triangles—taken together—approximate the arc BD. So $b_n n$ approximates the length of arc BD. The area of each inscribed triangle is $\frac{1}{2}h_n b_n$. Taken together, these n triangles approximate the sector BCD. Let $A_n = \left(\frac{1}{2}h_n b_n\right) n = \frac{1}{2}h_n(b_n n)$ be the sum of the areas of all n triangles. So A_n approximates the area of the sector BCD. When n is taken to be sequentially larger and larger, two things happen simultaneously:

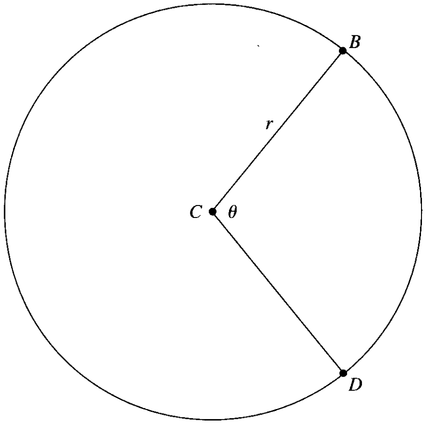


Figure 3.14

(1) The numbers h_n close in on the radius r, and the numbers $b_n n$ on arc BD. As a result, the numbers $\frac{1}{2}h_n(b_n n)$ close in on $\frac{1}{2}r(\text{arc}\,BD)$.

(2) The numbers A_n close in on the area of the sector BCD.

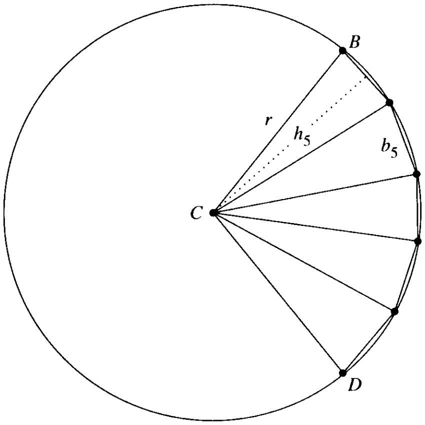


Figure 3.15

Because the numbers $A_n = \frac{1}{2}h_n(b_n n)$ can close in on one number only, it follows that the area of the sector BCD must be equal to $\frac{1}{2}r(\text{arc}\,BD)$. In the notation of limits, we can abbreviate what was just asserted by writing

$$\text{Area of sector } BCD = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{1}{2}h_n(b_n n)$$
$$= \frac{1}{2}r(\text{arc}\,BD).$$

Since we know that $\theta = \frac{\text{arc}\,BD}{r}$ radians, we have now established the fact that the area A of a sector of radius r and angle θ (given in radians) is equal to

$$A = \frac{1}{2}r^2\theta.$$

With $\theta = 2\pi$, this gives us the familiar expression πr^2 for the area of a circle of radius r.

3.3 Playing with Squares

Consider a square with side of length a. Of course, the area A of the square is $A = a^2$. Nothing else needs to be said. Nevertheless, we will now consider a completely different approach to this area. It may seem completely idiotic, but it will pay dividends later.

Subdivide the square into four equal squares and color three of them black, as shown in Figure 3.16. Denote the area of the resulting black "L" by $\mathbf{B}$. (Note that $\mathbf{B} = \frac{3}{4}A$, but this is irrelevent for now.) Do the same thing with the remaining white square in the upper right corner. Since each of these subdividing squares has area one-fourth that of the original white square, it follows that this smaller black L has area $\frac{1}{4}\mathbf{B}$; see Figure 3.17. Repeating this construction again produces a black L, this time of area $\frac{1}{4}\left(\frac{1}{4}\mathbf{B}\right) = \left(\frac{1}{4^2}\right)\mathbf{B}$, and so on.

Now consider the following steps. Step 0 is the creation of the black area $\mathbf{B}$. Step 1 adds the black L of area $\frac{1}{4}\mathbf{B}$ to $\mathbf{B}$ and produces the black area $\mathbf{B} + \frac{1}{4}\mathbf{B}$. Step 2 adds the next black L and produces the black area $\mathbf{B} + \frac{1}{4}\mathbf{B} + \frac{1}{4^2}\mathbf{B}$. Step 3 adds the next black L of area $\frac{1}{4}\left(\frac{1}{4^2}\mathbf{B}\right) = \frac{1}{4^3}\mathbf{B}$, and so on. The results of Steps 0 through 3 are illustrated in Figure 3.18. They produce the black areas

$$\mathbf{B},\quad \mathbf{B}+\frac{1}{4}\mathbf{B},\quad \mathbf{B}+\frac{1}{4}\mathbf{B}+\frac{1}{4^2}\mathbf{B},\quad \mathbf{B}+\frac{1}{4}\mathbf{B}+\frac{1}{4^2}\mathbf{B}+\frac{1}{4^3}\mathbf{B}.$$

This construction can be continued indefinitely. Make note of the emerging pattern and observe that for any number n (possibly very large), Step n creates the black area

$$\mathbf{B} + \frac{1}{4}\mathbf{B} + \frac{1}{4^2}\mathbf{B} + \frac{1}{4^3}\mathbf{B} + \cdots + \frac{1}{4^n}\mathbf{B}.$$

Each step produces an approximation of the area A of the original square. The area of the small white square that remains in the upper right corner is the error of the approximation. Let's compute these errors. After Step 0, the area of the small white square is $\frac{1}{4}A$. After Step 1, it is one-fourth of this, or $\frac{1}{4^2}A$. After Step 2, it is one-fourth that of Step 1, or $\frac{1}{4^3}A$. And after Step 3, it is $\frac{1}{4^4}A$. Continue the pattern, and observe that the area of the white square after Step n is $\frac{1}{4^{n+1}}A$. For example, the error of Step 14 is $\frac{1}{4^{15}}$ (and hence less than 0.000000001) of the area A. Since $\frac{1}{4^{n+2}}$

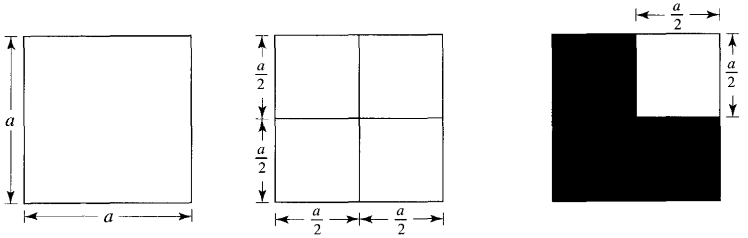
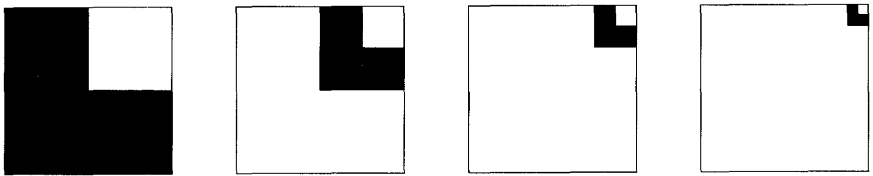
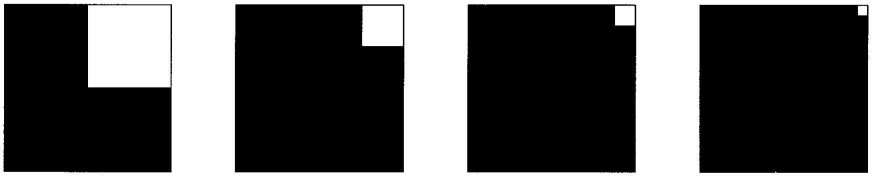


Figure 3.16



$\mathbf{B}$ $\quad \frac{1}{4}\mathbf{B}$ $\quad \left(\frac{1}{4}\right)^2\mathbf{B}$ $\quad \left(\frac{1}{4}\right)^3\mathbf{B}$

Figure 3.17



Step 0 Step 1 Step 2 Step 3

Figure 3.18

is less than $\frac{1}{4^{n+1}}$, it follows that the error of each step is less than the preceeding error.

In summary, for any n, Step n produces the partition

$$A = \left(\mathbf{B} + \frac{1}{4}\mathbf{B} + \frac{1}{4^2}\mathbf{B} + \cdots + \frac{1}{4^n}\mathbf{B}\right) + \frac{1}{4^{n+1}}A$$

of the area of the original square into a black area and a white area. The black area is an approximation of A, and the white area $\frac{1}{4^{n+1}}A$ is the error of the approximation. The approximation can be made as accurate as we need to have it. Simply choose n large enough so that the error $\frac{1}{4^{n+1}}A$ is small enough!

In the present case of the square, all this is rather superfluous because we already know that its area is $A = a^2$. The point is, however, that this strategy can be applied to compute the areas of regions that are not known beforehand. We will see shortly how Archimedes used it to compute the area of a parabolic section.

We conclude with a consequence of our analysis. By Figure 3.16, $\mathbf{B} = \frac{3}{4}A$; so $A = \frac{4}{3}\mathbf{B}$. Substituting this into the preceding equation for A gives

$$\frac{4}{3}\mathbf{B} = \left(\mathbf{B} + \frac{1}{4}\mathbf{B} + \frac{1}{4^2}\mathbf{B} + \cdots + \frac{1}{4^n}\mathbf{B}\right) + \left(\frac{1}{4^{n+1}} \cdot \frac{4}{3}\mathbf{B}\right).$$

After canceling **B**, we get $\frac{4}{3} = 1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^n} + \frac{1}{4^n} \cdot \frac{1}{3}$. Thus, for any positive integer n,

$$1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^n + \frac{1}{3}\left(\frac{1}{4}\right)^n = \frac{4}{3}.$$

This is an identity established by Archimedes. Note that $1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^n = \frac{4}{3} - \frac{1}{3}\left(\frac{1}{4}\right)^n$. Note also that for larger and larger n, the term $\frac{1}{3}\left(\frac{1}{4}\right)^n$ gets closer and closer to 0. Therefore, the numbers

$$1 + \tfrac{1}{4}$$

$$1 + \tfrac{1}{4} + \left(\tfrac{1}{4}\right)^2$$

$$1 + \tfrac{1}{4} + \left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{4}\right)^3$$

$$\vdots$$

$$1 + \tfrac{1}{4} + \left(\tfrac{1}{4}\right)^2 + \cdots + \left(\tfrac{1}{4}\right)^n$$

$$\vdots$$

close in on $\frac{4}{3}$. We abbreviate this observation by writing

$$\lim_{n\to\infty}\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^n\right) = \frac{4}{3}.$$

Let's rewrite the preceeding sum more compactly:

$$\sum_{i=0}^{n}\left(\frac{1}{4}\right)^i = 1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^n.$$

This "sigma notation" (Σ is the capital Greek sigma, i.e., "S," so it suggests "Sum") was not used by Archimedes. It was introduced only in the 18th century by the Swiss mathematician Leonhard Euler. The stategy implicit in it is this: The first i is $i = 0$. Plug it into $\left(\frac{1}{4}\right)^i$ to get $\left(\frac{1}{4}\right)^0 = 1$ and write $+$. Next, put in what you get by letting $i = 1$, namely $\left(\frac{1}{4}\right)^1$ and write $+$. Continue until the last i, namely $i = n$, is reached. In a similar way, when $n = 6$,

$$\sum_{i=3}^{6} 2^i = 2^3 + 2^4 + 2^5 + 2^6 = 8 + 16 + 32 + 64 = 120$$

With this shorthand notation the limit equation can be rewritten $\lim_{n\to\infty}\sum_{i=0}^{n}\left(\frac{1}{4}\right)^i = \frac{4}{3}$.

3.4 The Area of a Parabolic Section

We now turn to Archimedes's work *Quadrature of the Parabola*, in which he computes the area of a section of a parabola. Quadrature refers to "squaring," and squaring a given region bounded by one or more curves refers to the classical problem of constructing a rectangular region with the same area as the given one.

Take any parabolic section SVS', where V is the vertex. (Refer back to Figure 3.3(a).) Inscribe the triangle $\Delta SVS'$, as shown in Figure 3.19. Archimedes's objective is the proof of the fact that the area of the parabolic section is equal to four-thirds the area of the inscribed triangle. Archimedes's argument consists of a number of steps, each carefully laid out and each rather routine. The argument as a whole, however, is formidable. Let's follow along.

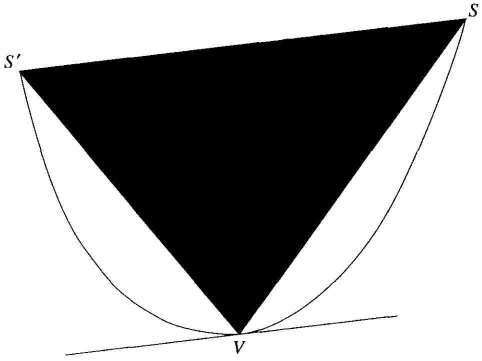


Figure 3.19

Refer to Figure 3.20. Let A be the midpoint of the segment $S'S$. Draw a line through S parallel to VA, and let its point of intersection with the tangent at V be E. Do a similar thing on the other side of VA.

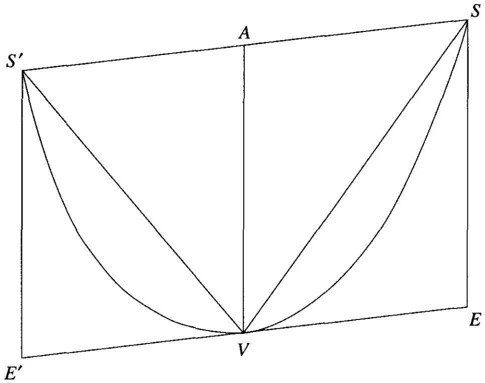


Figure 3.20

Next, Archimedes takes B to be the midpoint of the segment VE and draws the segment BD parallel to VA. (See Figure 3.21.) The point C is the intersection of BD with the parabola, and G is the intersection of BD and VS. He does the same thing on the other side.

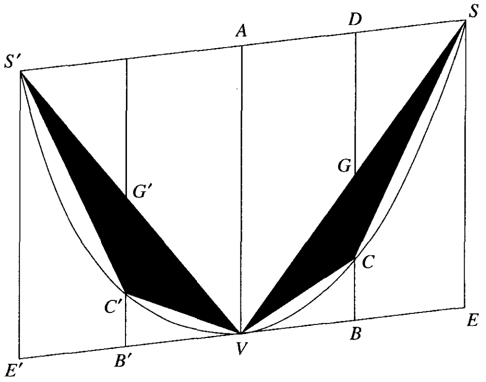


Figure 3.21

The key step in the argument—focus on the right side of the figure—is now this:

Proposition A. The area of the triangle ΔVAS is equal to four times the area of the triangle ΔVCS.

Archimedes argues as follows. Because B is the midpoint of VE, it follows that $\frac{VB}{VE} = \frac{1}{2}$. So by Proposition 3.3 of Section 3.1, $\frac{BC}{ES} = \frac{VB^2}{VE^2} = \frac{1}{4}$. Since $ES = BD$, he gets

$$BC = \frac{1}{4}BD.$$

Next consider Figure 3.22, which is extracted from Figure 3.21. By similar triangles, $\frac{BG}{ES} = \frac{VB}{VE} = \frac{1}{2}$. Since $ES = BD$, it follows that $BG = \frac{1}{2}BD$. So G is the midpoint of BD, and hence $BG = GD$. Because $BG = \frac{1}{2}BD$, it follows that $\frac{1}{2}BG = \frac{1}{4}BD$. But $\frac{1}{4}BD = BC$, so $BC = \frac{1}{2}BG$. Therefore, C is the midpoint of BG, and $CG = BC$. By a substitution,

$$CG = \frac{1}{2}GD.$$

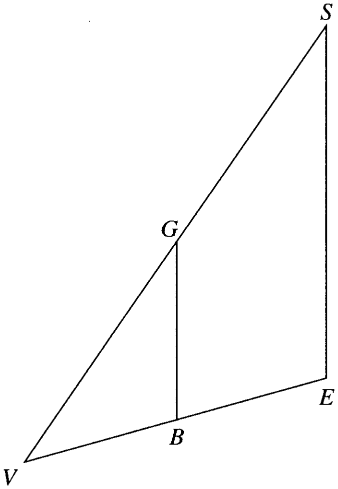


Figure 3.22

Figure 3.23 consists of two figures extracted from Figure 3.21. Concentrate on the two triangles at the top. Note that they have the same height. Since $CG = \frac{1}{2}GD$, it follows that

$$\text{Area}\,\Delta DGS = 2\,\text{Area}\,\Delta GCS.$$

Concentrating on the bottom figure, we get in the same way that Area $\Delta DGV = 2$ Area ΔGCV. Adding areas,

$$\text{Area}\,\Delta DGS + \text{Area}\,\Delta DGV$$
$$= 2(\text{Area}\,\Delta GCS + \text{Area}\,\Delta GCV).$$

Refer back to Figure 3.21 and notice that this equality says that Area $\Delta DVS = 2$ Area VCS. Refer to Figure 3.21 once more and observe that Area $\Delta DVA =$ Area ΔDVS. (Why?) It follows that Area $\Delta VAS = 2$ Area ΔDVS. Combining equalities gives

$$\text{Area}\,\Delta VAS = 4\,\text{Area}\,\Delta VCS$$

as required in Proposition A.

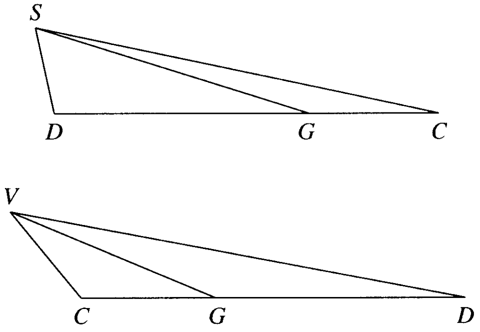


Figure 3.23

The same argument applied to the left side of Figure 3.21 shows that

$$\text{Area}\,\Delta VAS' = 4\,\text{Area}\,\Delta VC'S'.$$

Therefore,

$$\text{Area}\,\Delta SVS' = 4(\text{Area}\,\Delta VCS + \text{Area}\Delta VC'S').$$

So Archimedes has verified

Proposition B. In reference to Figure 3.21,

$$\text{Area}\,\Delta VCS + \text{Area}\,\Delta VC'S' = \frac{1}{4}\,\text{Area}\,\Delta SVS'.$$

The brilliance of Archimedes's argument becomes apparent now: Since B and B' are the midpoints of VE and VE', respectively, Proposition 3.2 of Section

3.1 applies and asserts that C and C' are the respective vertices of the parabolic sections VCS and $VC'S'$. Consider now the two parabolic sections VCS and $VC'S'$ with their circumscribed parallelograms; see Figure 3.24. Applying Proposition B to each of them shows that the sum of the areas of the two black triangular regions on the right is $\frac{1}{4}$ Area ΔVCS and that of the two black triangles on the left is $\frac{1}{4}$ Area $\Delta VC'S'$. Since we have already seen that

$$\text{Area}\,\Delta VCS + \text{Area}\,\Delta VC'S' = \frac{1}{4}\,\text{Area}\,\Delta SVS',$$

the four black triangles together have area

$$\left(\frac{1}{4}\right)^2 \text{Area}\,\Delta SVS'.$$

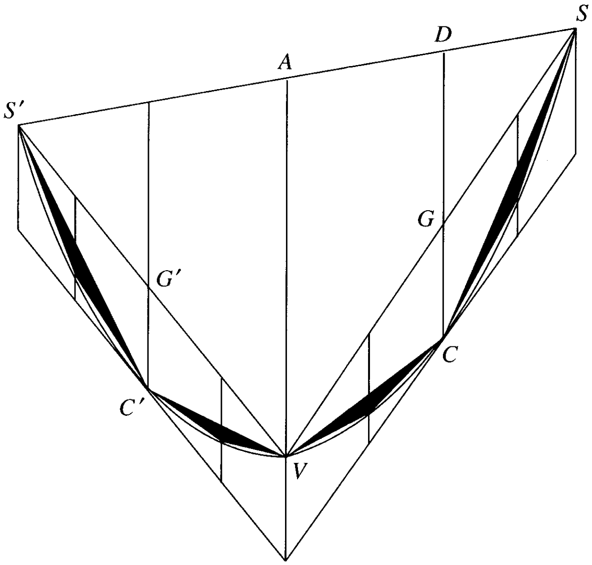


Figure 3.24

Let $\mathbf{B} = \text{Area}\,\Delta SVS'$. Recall that the black area of Figure 3.21 is $\frac{1}{4}\mathbf{B}$, and that of Figure 3.24 is $\left(\frac{1}{4}\right)^2\mathbf{B}$. Archimedes's procedure of forming triangles can be repeated again and again. The principle is now the same as that in the approximation of the area of the square in Section 3.3. When the black areas produced by each step are taken together, they provide an approximation of the area of the parabolic section. The first three approximations, obtained by adding the black regions already determined (those of Figures 3.19, 3.21, and 3.24), are shown in Figure 3.25. Observe the pattern, and notice that when the areas of the triangles produced after n steps are added together, an approximating inscribed region of area

$$A_n = \left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots + \left(\frac{1}{4}\right)^n\right)\mathbf{B}$$

is obtained. Now let n get larger and larger, and recall from Section 3.3 that

$$\lim_{n\to\infty}\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots + \left(\frac{1}{4}\right)^n\right) = \frac{4}{3}.$$

Because the numbers A_n close in on

$$\frac{4}{3}\mathbf{B} = \frac{4}{3}\,\text{Area}\Delta SVS',$$

it follows that this is the area of the parabolic section SVS'. The proof of Archimedes's theorem is now complete:

Archimedes's Theorem. The area of the parabolic section SVS' is equal to four-thirds the area of the inscribed triangle SVS'.

We have witnessed a remarkable argument. Its mathematical precision is decidedly superior to most of what would follow over the next 1800 years! In its last step (the "limit" step), the proof you saw differs from Archimedes's original. The Greeks did not consider (or perhaps, more accurately, did not accept as legitimate) arguments that used "infinite" processes. Instead, Archimedes let A be the area of the parabolic section SVS' and $A' = \frac{4}{3}$ Area $\Delta SVS'$, and he demonstrated the equality $A = A'$ by showing that $A > A'$ and $A < A'$ are both impossible. About 2000 years later, arguments involving infinite processes become the heart and soul of integral calculus; however, mathematicians wrestled with their legitimacy until the 19th century.

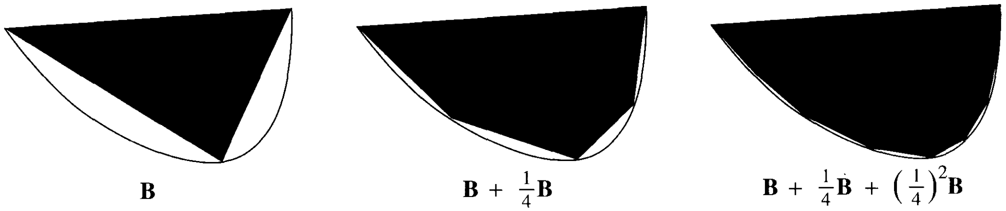


Figure 3.25

3.5 The *Method*

There were references in the transmitted classical literature to a certain treatise of Archimedes called *Method*. The work itself, unfortunately, appeared to have been lost. However, in the early 1900s a Danish historian learned of a certain catalogue describing the holdings of the libraries of the Greek Orthodox Church. The catalogue listed a document that contained—in two superimposed layers—both a newer and an older manuscript, the latter of a mathematical nature. He traveled to Constantinople in the summer of 1906 and found a document consisting of about two hundred leaves of parchment. It contained Greek Orthodox prayers from the 13th and 14th centuries, and indeed, most of the leaves showed a more or less distinct underlayer: a mathematical manuscript in 10th century notation! Most of it was readable. A careful study revealed that it contained work of Archimedes. For the most part it was work already known, but among the new things there was the *Method*. This was an important find. It gave insights into the method that Archimedes had used to discover the connections between various areas and volumes and, in particular, between the areas of the parabolic section and the inscribed triangle.

The essence of his method is this: geometrical regions are compared by slicing them into thin strips and placing these (surprise!) at the two ends of a lever. To illustrate it, we return to the parabolic section.

Refer again to Figure 3.21 and extend the segment VC so that it intersects ES at F. (See Figure 3.26.) By Proposition A,

$$\text{Area}\,\Delta VAS = 4\ \text{Area}\,\Delta VCS.$$

Since the triangles VAS and VES are identical, their areas are the same. Therefore,

$$\text{Area}\,\Delta VES = 4\,\text{Area}\,\Delta VCS.$$

Recall from the proof of Archimedes's theorem that B is the midpoint of VE and C is the midpoint of BG. Since ΔVEF is similar to ΔVBC, we have $\frac{BC}{EF} = \frac{1}{2}$, or $EF = 2BC$. Because C is the midpoint of BG, it follows that $2BC = BG$. Therefore, $BG = EF$. Now, ΔVES is similar to ΔVBG and B is the midpoint of VE, so $\frac{BG}{ES} = \frac{1}{2}$. Hence $ES = 2BG = 2EF$, and it follows that F is the midpoint of ES.

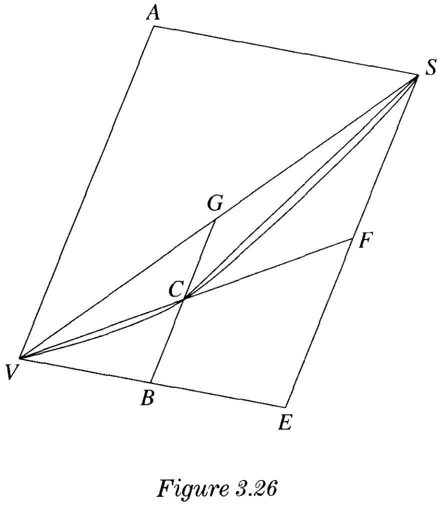


Figure 3.26

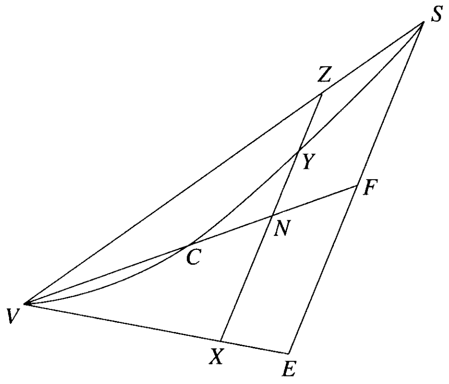


Figure 3.27

Now let X be an arbitrary point on VE and draw XZ parallel to ES. (See Figure 3.27.) Let N and Y be the indicated points of intersection. Archimedes next proceeded to the important step of verifying the equality

$$XZ \cdot NF = YZ \cdot VF.$$

By similar triangles, $\frac{VN}{VX} = \frac{VF}{VE}$, and therefore $VN = \frac{VX \cdot VF}{VE}$. Also by similar triangles, $\frac{XZ}{VX} = \frac{ES}{VE}$, and hence $XZ = \frac{VX \cdot ES}{VE}$. By an application of Proposition 3.3 of Section 3.1, $\frac{XY}{ES} = \frac{VX^2}{VE^2}$. By combining these equalities,

$$\begin{aligned} VN \cdot XZ &= \left(\frac{VX \cdot VF}{VE}\right)\left(\frac{VX \cdot ES}{VE}\right) \\ &= \frac{VX^2}{VE^2} \cdot VF \cdot ES = \frac{XY}{ES} \cdot VF \cdot ES \\ &= XY \cdot VF. \end{aligned}$$

So by Figure 3.27, $(VF - NF) \cdot XZ = VN \cdot XZ = XY \cdot VF = (XZ - YZ) \cdot VF$. After multiplying out and subtracting, $-XZ \cdot NF = -YZ \cdot VF$. The equality $XZ \cdot NF = YZ \cdot VF$ is now verified.

Now think of both the triangle ΔVES and the parabolic section VCS (Figure 3.26) to be made out of very thin, rigid (no bending), and homogeneous (no lumps) material. Imagine ΔVES being sliced up into very thin strips, as shown in Figure 3.28. To every strip XZ of ΔVES there corresponds—by Figure 3.27—the strip YZ of the parabolic section VCS. In this way, the parabolic section is sliced into strips also. Choose a unit of weight in such a way that a strip 1 unit long weighs 1 unit. So the weight of any strip is numerically equal to its length. Again refer to Figure 3.27 and extend the segment VF to a point K such that $FK = VF$; see Figure 3.29. Consider the segment VFK to be a lever with its fulcrum at F. Take any strip XZ of ΔVES and the corresponding strip YZ of the parabolic section. Refer to the Exercises 3C for the law of the lever, and notice that the equality $XZ \cdot NF = YZ \cdot VF$ means that each strip XZ balances the strip YZ placed at K. Doing this for each strip of ΔVES shows that ΔVES is balanced by the sum total of all of the strips of the parabolic section VCS placed at K.

Observe next that the product weight × distance for the right side of the lever is equal to

$$\begin{aligned} &(\text{Area of parabolic section } VCS) \cdot FK \\ &\quad = (\text{Area of parabolic section } VCS) \cdot VF. \end{aligned}$$

Now consider the left side of the lever. In his work *On the Equilibrium of Planes*, Archimedes showed that the *center of mass* (also called the *centroid*) of the triangle ΔVES is the point M on VF with $MF = \frac{1}{3}VF$. (See Exercise 13.) Considering the entire weight of ΔVES to be concentrated at M, the product weight × distance for the left side of the lever is

$$(\text{Area } \Delta VES) \cdot MF = (\text{Area } \Delta VES) \cdot \frac{1}{3} VF.$$

Since the system is in balance, Archimedes has now established that

$$\begin{aligned} &(\text{Area of parabolic section } VCS) \cdot VF \\ &\quad = (\text{Area } \Delta VES) \cdot \frac{1}{3} VF. \end{aligned}$$

Canceling VF and using the fact (noted earlier) that Area $\Delta VES = 4$ Area ΔVCS finally shows that

$$\text{Area of parabolic section } VCS = \frac{4}{3} \text{ Area } \Delta VCS.$$

Noting finally that C is the vertex of the parabolic section VCS (by Proposition 3.2 of Section 3.1), we

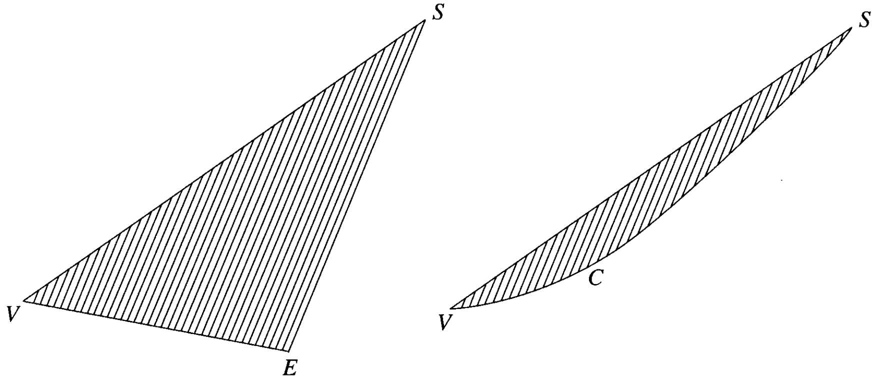


Figure 3.28

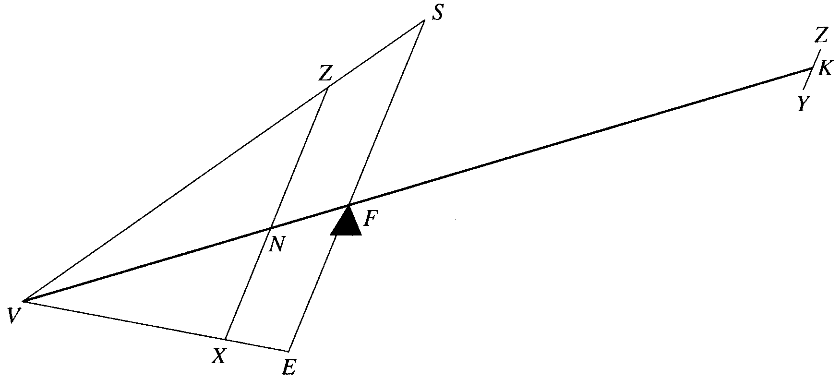


Figure 3.29

now see that Archimedes has established his theorem again, this time for the parabolic section VCS.

3.6 Postscript

We have taken a glimpse at the mathematical genius of Archimedes. When Plutarch praised "the precision and cogency of the methods and means of proof" of Archimedes's studies, we can now understand what he meant. When Archimedes's investigations (and also those of Euclid, Aristarchus, Apollonius, Ptolemy, and the other classical scholars) were appreciated and understood, an important source of light began to displace some of the darkness of the Middle Ages. In reference to mathematics, there can be little doubt that Archimedes's analysis of the parabola influenced the development of integral calculus in the 17th century. Of course, since it was rediscovered only in 1906, the influence of the *Method* could have been only indirect.

Exercises
3A. The Circle and Related Areas

1. Consider a triangle that is inscribed in a circle and has a diagonal of the circle as one of its sides. Use Proposition 2.1 of Section 2.2 to show that this triangle is a

right triangle. Show this again, by using Ptolemy's theorem and Pythagoras's theorem.

2. Compute the area of the circular sector of radius 7 and angle $\frac{\pi}{5}$. Then compute the area of the circular sector of radius 5 and an arc of length 8.

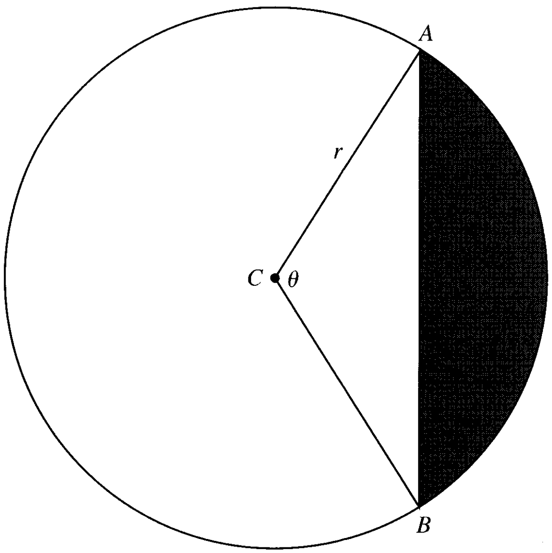


Figure 3.30

3. Show that the shaded area of the circle in Figure 3.30 is equal to $\frac{1}{2}\theta r^2 - r^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2}$.
4. Compute the section of the circle in Exercise 3 if $r = 5$ and $\theta = 50^\circ$.
5. In the diagram of Figure 3.31, O is the center of a circle and the radii AO and OB meet at O at right angles. The second arc connecting A and B is half of the circle with diameter AB and center C. Show that the area of the moon-shaped region—such regions are called *lunes*[2]—produced by the two circular arcs is equal to the area of the triangle AOB.
6. Suppose that the circle of Exercise 5 has radius r. Show that the area of the lune is $\frac{1}{2}r^2$ and that of the section of the circle between the lune and $\triangle AOB$ is equal to $\frac{\pi}{4}r^2 - \frac{1}{2}r^2$.

3B. Sigma Notation and Areas

7. Write the sums $1+2+3+4$; $1+2^2+3^2+4^2+5^2$; and $1+2^2+3^3+4^4+5^5+6^6$ using sigma notation.

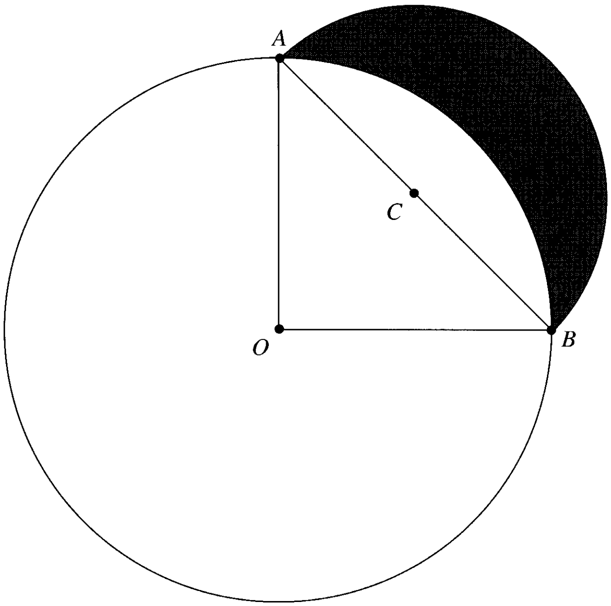


Figure 3.31

8. The squares in Figure 3.32 are each of side 1. Continue the indicated pattern to show that $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{2}\right)^i = 1$.

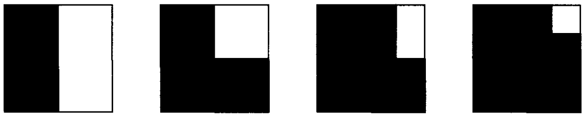


Figure 3.32

9. Consider a triangle with base b and height h. Refer to Figure 3.33. Inscribe into the triangle the rectangles R_1, R_2, and R_2', where the base of R_1 is $\frac{1}{2}b$, and the bases of R_2 and R_2' are $\frac{1}{2}\left(\frac{1}{2}b\right) = \frac{1}{4}b$; and continue the pattern indicated. Verify that the area of this triangle is $\frac{1}{2}bh$ by filling the triangle with this sequence of rectangles. In other words, show that

$$\frac{1}{2}bh = \text{Area } R_1 + (\text{Area } R_2 + \text{Area } R_2') + \ldots.$$

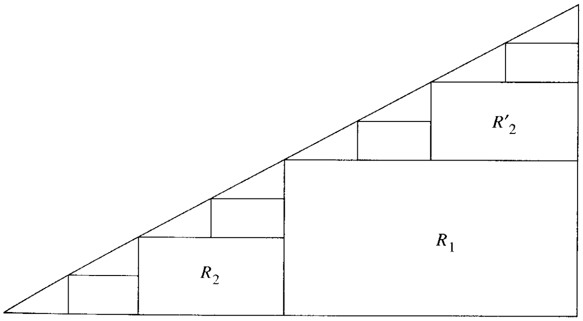


Figure 3.33

10. Consider a parabolic section where the cut is 7 units long and the perpendicular distance from the cut to the vertex is 4 units. What is the area of the section?

3C. Archimedes's Law of the Lever

A lever consists of a rigid arm and a pivot called a *fulcrum*. The arm is considered to be weightless (in practical situations it would be of negligible weight). In Figure 3.34 the arm is represented by the segment through the points P_1 and P_2 and the fulcrum is positioned at F. If two point-masses of weights m_1 and m_2 are placed on the lever at points P_1 and P_2 in such a way that $m_1 d_1 = m_2 d_2$, where d_1 is the distance from P_1 to F and d_2 is the distance from P_2 to F, then the system balances, i.e., it is *in equilibrium*. This is Archimedes's law of the lever. Notice that a great weight m_1 can be balanced by a small weight m_2 provided that d_2 is large enough. The lever was most certainly used by the Egyptian pyramid builders. Archimedes is to have said (in reference to the lever), "Give me a place to stand on and I will move the Earth."

11. Suppose $d_1 = 1$ and $m_1 = 2000$. If $d_2 = 10$, what weight has to be placed at P_2 to raise m_1?

12. Determine the center of mass of the system consisting of the two masses shown in Figure 3.35. The bigger block has a mass of 80 kilograms and the smaller a mass of 15 kilograms. The distance between the centers of the blocks is 9 meters.

3D. Facts Needed in the *Method*

An important step in Archimedes's argument concerns the center of mass (the centroid) of the triangle.

13. In the triangle ΔVES of Figure 3.36, F and F' are the midpoints of the segments ES and SV, respectively. Show that the areas of ΔVFE and ΔVFS are the same [Hint: they have equal bases; do they have equal heights?] Deduce that the centroid M must lie on the segment VF. By the same argument M must lie on EF'. So M is the intersection of VF and EF'.

14. Now extend the segment EF' of Figure 3.36 to a segment $EF'J$ in such a way that VJ is parallel to ES. Show that ΔFME and ΔVMJ are similar and that $\Delta ESF'$ and $\Delta JVF'$ are similar. Deduce that $FM = \frac{1}{2}VM$, and hence that $FM = \frac{1}{3}VF$, as proved by Archimedes.

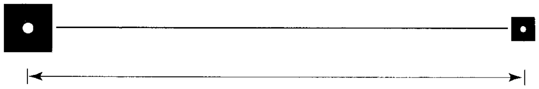


Figure 3.35

15. Refer to the proof of Archimedes's theorem in Section 3.5. Show that the verification of the equality $XZ \cdot NF = YZ \cdot VF$ is unaffected when XZ is placed to the left of C, in other words, when Figure 3.27 is replaced by Figure 3.37.

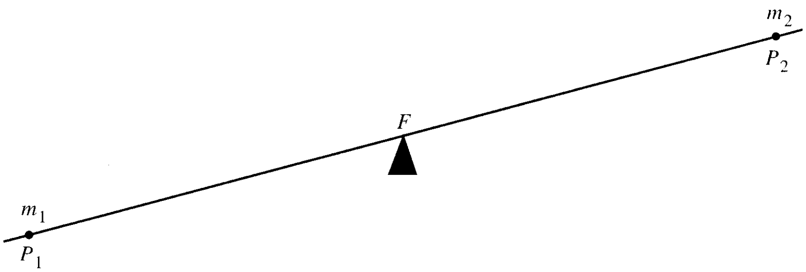


Figure 3.34

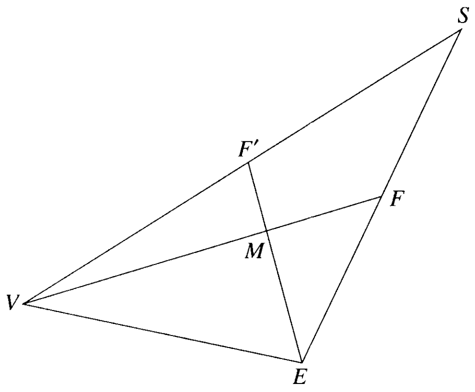


Figure 3.36

3E. Some Mathematics from the Middle Ages

16. The mathematician Gerbert (later Pope Sylvester II, 999–1003 A.D.) solved the following problem considered very difficult at the time. Let x and y be the legs of a right triangle and express x and y in terms of the height h and the area A of the triangle.

17. Gerbert expressed the area of an equilateral triangle of side a as $\left(\frac{a}{2}\right)\left(a-\frac{a}{7}\right)$. Show that this is an approximation that is equivalent to the approximation $\sqrt{3} \approx \frac{12}{7}$. Use a calculator to check its accuracy.

Note that the problems studied by Gerbert are much less sophisticated than those considered by the Greeks.

3F. Mathematics in the 16th Century

The 16th century saw great strides in the development of algebraic symbolism and notation. For example, the French mathematician François Viète used A to denote the unknown, A *quadratum* to denote A^2, and A *cubum* for A^3. He also used "+" and "−" the way we do, but he had no symbol for "=". What is now written $5bx^2 - 2cx + x^3 = d$, Viète would have written

$b5$ in A quad $-c$ plano 2 in $A+A$ cubum aequatum d solido.

So *aequatum* is =. It had been a common practice earlier to work with the "vernacular." In other words, the problem would have been written out: "Take a quantity times itself and multiply this by five times b, subtract from this two times c times the quantity …" and then it would have been solved without the algebraic manipulations that we use today!!

The 16th century also saw work on the solutions of cubic and quartic equations, e.g., equations of the form $x^3 + 3x - 6 = 0$ and $x^4 + 3x^2 + 5x - 8 = 0$. In fact, there were public competitions and challenges in Italy between some very interesting characters (Tartaglia, Cardano, Ferrari—not of sports car fame) to determine who was best at the game of solving such equations. (This required lots of ingenuity.) Instead of going into the details, we point out a related fact:

Let $p(x)$ be a polynomial. Recall that a number d such that $p(d) = 0$ is called a *root* of $p(x)$. Note that if $p(x)$ has a factorization of the form $p(x) = (x - d)q(x)$, then d is a root of $p(x)$. This fact works also in reverse: namely, if d is a root of $p(x)$, then $x - d$ divides $p(x)$.

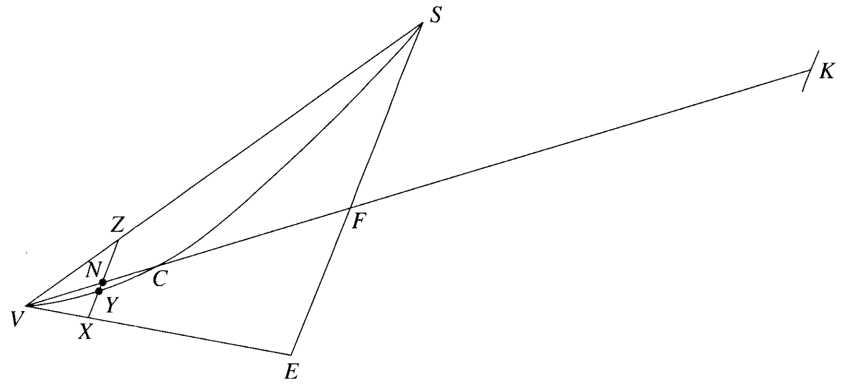


Figure 3.37

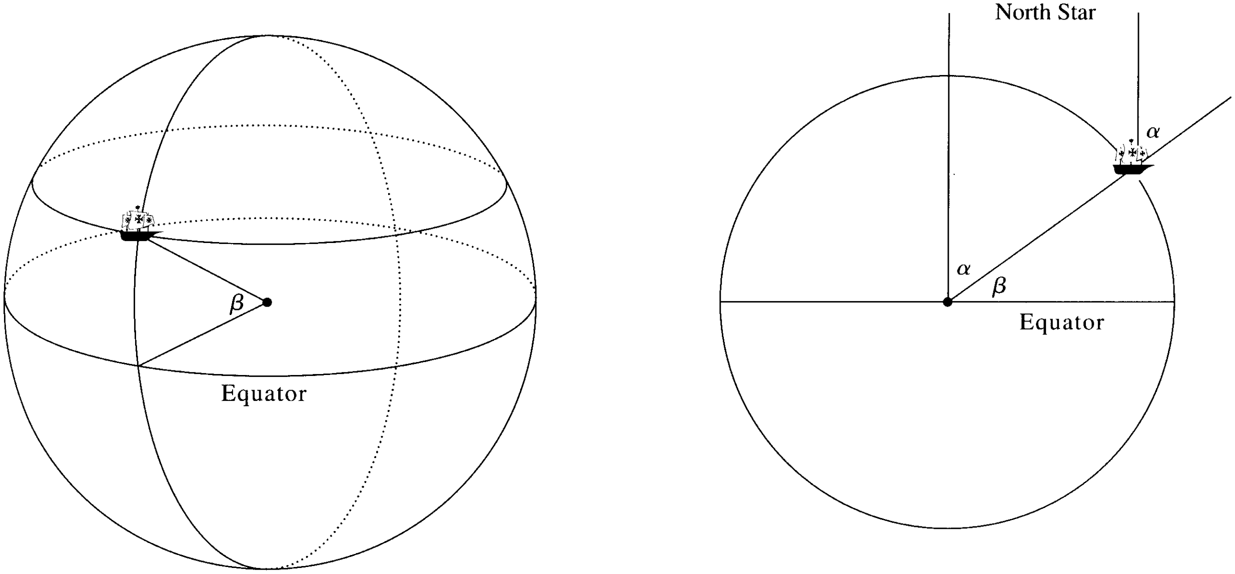


Figure 3.38

18. Use this idea to factor the polynomials $x^3 + 3x - 4$ and $x^3 - 7x - 6$ as completely as possible.

3G. Celestial Navigation

Elementary celestial navigation was common in the 16th century. The considerations of Eratosthenes and a star catalogue were used to determine the position—at least, the latitude—of a ship at sea.

19. Suppose a ship is positioned on the globe, as shown on the left in Figure 3.38. Note that the angle β determines its north-south position. This is the ship's *latitude*. As the diagram on the right shows, β can be determined if one knows α. But α can be measured: it is the angle between a plumb line and the line of sight to the North Star. Express the distance from the ship to the equator (along the arc) in terms of α and the radius of the Earth.

20. Suppose that the angle α is measured to be 53°. How many miles is the ship north of the equator? (Use the modern value of 3950 miles for the radius of the Earth.)

Why doesn't this strategy work for determining the *longitude*, i.e., the east-west position, of a ship? Without the solution of this "problem of longitude," a ship's position cannot be fully determined. This problem turned out to be very elusive and was not solved until the 18th century. Large cash prizes were offered for a solution by the governments of seafaring states like Spain, Portugal, Venice, Holland, and England. The British "Board of Longitude" offered the largest amount, a sum of 20,000 pounds. Note that as the Earth turns, the stars travel across the sky at a rate of 360° per day, or 15° per hour. Thus, if time can be measured accurately aboard a ship, the sky will provide its position. A ship's rolling motion rendered pendulum clocks useless. The issue, therefore, centered on the construction of a clock accurate under the severe conditions on board. The matter was not settled until 1761–1762. A marine chronometer devised by the Englishman John Harrison (it had taken him nineteen years to complete) was tested aboard one of His Majesty's ships and found to have lost only about 5 seconds in eighty days at sea.[3]

Notes

[1] Ptolemy's *Geographia*, translated into Latin at the beginning of the 15th century, contains geographical coordinates of 8000 localities. It laid the foundation of the science of cartography and was thus of direct importance to navigation in the 16th century. Its map of the known world, in particular of Europe and North-

ern Africa, is astounding. Columbus used Ptolemy's map (in part because its dimensions were a little too small) as evidence for his contention that it would be feasible to reach Asia by sailing west.

[2] Apparently, Leonardo da Vinci was fascinated by lunes. See J.L. Coolidge, *The Mathematics of Great Amateurs*, Clarendon Press, a Division of Oxford University Press, 1991.

[3] The interested reader may wish to engage Dava Sobel's *Longitude: The True Story of a Lone Genius Who Solved the Greatest Scientific Problem of His Time.* Walker, New York, 1995.

4

A New Astronomy and a New Geometry

The achievements of the Greeks—you are now in a position to judge for yourself—are remarkable. They developed trigonometry and geometry and applied both in spectacular ways to estimate the sizes and distances of objects in the universe and to create mathematical models that simulate the motion of these objects. They also developed subtle mathematical methods for computing areas and volumes, and they contributed to number theory and algebra. Given what we know now, however, it is clear that important elements were missing. The Greeks did not have the coordinatized line, and the correspondence between points and real numbers that it implies. Since they had no comprehensive coordinatized geometry, they could not make use of the powerful interaction between algebra and geometry that it implies. In summary, one can say that Greek mathematics is fundamental, substantial, and also brilliant,[1] but that it is incomplete. As for Greek physics and astronomy, however, the matter was different: the basic points of view were flawed and the underlying principles were in error. In time, the scholars of Europe absorbed the learning of Greece and reflected about its science. When they started to question and challenge it, a scientific revolution had begun.

4.1 A New Astronomy

Copernicus (1473–1543) was born in the Polish town of Torun. He studied astronomy in Bologna and medicine in Padua. After returning to Poland, he settled into his position as canon of the Church. He found ample time to pursue astronomy in his little study, which is said to have been in the tower of the cathedral. He was not satisfied with the very complicated nature of Ptolemy's astronomy and so published his own comprehensive theory of the solar system, *De Revolutionibus Orbium Celestium*. It placed the Sun at the center of the universe and the Earth and the other planets in orbit around it. The introduction of an early edition of *De Revolutionibus* makes mention of Aristarchus: "...some say that Aristarchus of Samos was of the same opinion." Later editions apparently dropped this reference. Realizing that it might be controversial (after all, in Genesis it is the Earth that is at the center of God's creation), Copernicus delayed the appearance of this work until 1543, the year of his death.

With the Sun at the center of the system, the retrograde motion of the planets no longer required an elaborate explanation. Therefore, Ptolemy's convoluted orbital clockwork of circles upon circles became unnecessary. However, Copernicus did retain basic elements of Ptolemy's mathematics. For instance, his model for the Earth's orbit was Ptolemy's off-centered circle. Furthermore, when discrepancies between his theory of simple circular orbits and observations persisted, Copernicus was later led back to the epicycloids of Ptolemy.

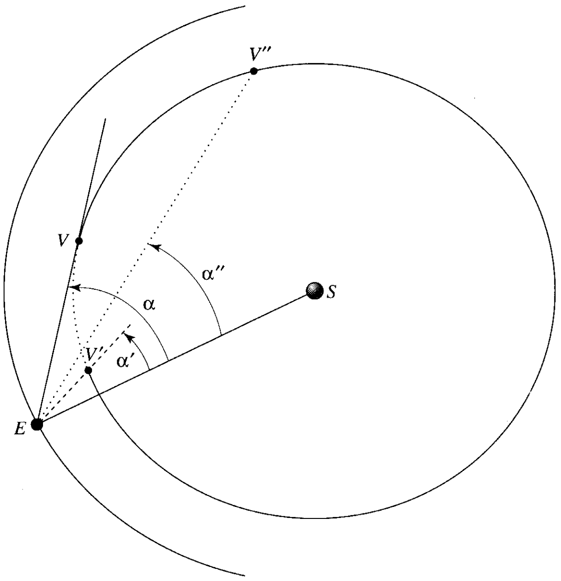


Figure 4.1

But there were also new mathematical elements in Copernicus's work. The planet Venus provides an example. Copernicus determined its distance from the Earth with the following argument. The angle α

between the lines of sight from an observer on Earth to the Sun on the one hand, and to Venus on the other, is greatest when the line of sight to Venus is tangent to its circular orbit around the Sun. (See Figure 4.1.) But then the angle between the line of sight to Venus and the radius VS of the orbit is a right angle. Therefore,

$$\sin\alpha = \frac{VS}{ES}.$$

The greatest value for α that Copernicus observed was $\alpha = 46°$. Since $\sin 46° = 0.719$ (recall that $\sin 45° = \frac{\sqrt{2}}{2} = 0.707$), he obtained

$$VS = 0.719\,ES.$$

Let's define the distance between the Earth and the Sun to be one astronomical unit (AU). So $ES = 1$ AU. Therefore, Copernicus has shown that the distance between Venus and the Sun is 0.719 AU. In a similar manner (this is more complicated in the case of the outer planets such as Mars), Copernicus was able to compute the distances in AUs between the Sun and all the planets that were known at the time. As you can see from Table 4.1, he did so quite accurately.

Table 4.1

Estimated Average Distances of the Planets from the Sun in Astronomical Units

	Copernicus	**Kepler**	**Modern**
Mercury	0.376	0.389	0.387
Venus	0.719	0.724	0.723
Earth	1.000	1.000	1.000
Mars	1.520	1.523	1.524
Jupiter	5.219	5.200	5.202
Saturn	9.174	9.510	9.539

The Danish astronomer Tycho Brahe (1546–1601) designed and constructed new calibrated precision instruments that allowed him to observe celestial phenomena with a much higher degree of accuracy than was previously possible. In 1572 he observed a very bright star in a location where there had been no star before. When this "nova stella" disappeared about a year and a half later, it was clear to him that the sphere of stars was not as immutable as the Greeks had claimed. (What Tycho had seen was no doubt a supernova, an exploding star. Ironically, it was apparently an explosion that also killed Tycho. It is said that his bladder burst after some heavy drinking.) In 1577 he observed a comet that appeared to have no problem moving through the crystal spheres that supposedly held the planets in their orbits. Tycho realized that this explanation of the Greeks was also in error. Tycho developed his own "Tychonic" system of the universe: the Moon and the Sun are in orbit about the Earth, but all other planets are in orbit about the Sun. For over a century, the Ptolemaic, Copernican, and Tychonic explanations of the universe existed side by side.

Johannes Kepler (1571–1630) was born near Stuttgart, in southern Germany. His father was a mercenary whose only pleasures seemed to have been drink and war. He deserted his family soon after Johannes Kepler's birth. Young Kepler was sickly and poor, but he was also a brilliant student. Being deeply religious, he enrolled in the University of Tübingen to study theology. While there, he learned about the principles of the Copernican system from a professor of mathematics (privately, since the university administration frowned upon such heretical studies). He graduated in 1591 and took the dual post of Mathematician of the Province and teacher at the Evangelical Seminary in Graz, Austria. He continued his speculations about astronomy, with particular interest in the relationship between the distances and velocities of the planets, as well as the driving forces that propel them in their orbits.

In the meantime, Tycho Brahe had accepted an appointment as astronomer to the royal court of the emperor of Austria in Prague. He brought with him some of his instruments as well as the wealth of accurate data that he had collected. When Kepler's work came to his attention, Tycho arranged for Kepler to become his assistant. After Tycho's death, Kepler was free to use Tycho's data in the pursuit of his own investigations and turned his attention to the orbit of Mars. Convinced of the correctness of Copernicus's system, he set out to verify that this orbit is a circle. A circle, not in the simple sense, but in the sense of Ptolemy; see Figure 2.26 of Section 2.6. It was Kepler's goal to determine for the orbit of Mars the precise positions of the Sun S (S replaces E in the figure just mentioned), the center C of the orbit, and the point of reference A for the velocity. He computed orbital distances and angles for many different positions for S, C, and A, and checked and rechecked his computations against Tycho's observations. But after three years of grueling work, he was forced to conclude that the orbit of Mars could *not* be a circle. He wrote:

> The planet's [Mars's] orbit is not a circle; but [starting at aphelion] it curves inward little by little and then [returns] to the amplitude of the circle at [perihelion]. An orbit of this kind is called an oval.[2]

Circles and spheres satisfied Kepler's sense of aesthetics, but the oval did not. "If only the orbit were an ellipse," he wrote, "then the problem would have been solved already by Archimedes and Apollonius." But an oval! After all his efforts he was left, as he put it, with "only a single cartful of dung."

Kepler discarded his work and started afresh. New calculations and comparisons with observation finally brought success. He concluded that the orbit of Mars curves inward at *both* aphelion and perihelion and—at last—that it has the shape of an ellipse. In 1609 he published his first two laws of planetary motion in the book *Astronomia Nova*.

1. Each planet P moves in an elliptical orbit with the Sun S at one of the focal points of the ellipse. (See Figure 4.2.)
2. A given planet sweeps out equal areas in equal times.

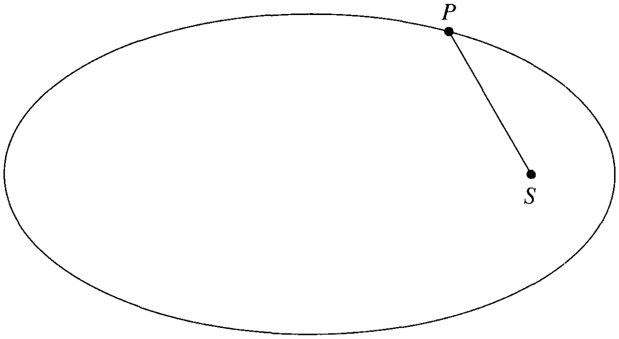


Figure 4.2

Stated more precisely, this second law says the following: Suppose that it takes a planet time t_1 to move from position P_1 to Q_1 in its orbit, and that it traces out the area A_1 in the process. See Figure 4.3. Suppose that it later moves from P_2 to Q_2 in time t_2 while tracing out the area A_2. According to Kepler's second law,

$$\text{if } t_1 = t_2, \text{ then } A_1 = A_2.$$

A moment's thought shows that the second law implies that a planet moves faster when it is closer to the Sun than when it is farther away.

The second law also implies that if $t_1 = t_2 + t_2$, then $A_1 = A_2 + A_2$. So if $t_1 = 2t_2$, then $A_1 = 2A_2$. Similarly, if $t_1 = 3t_2$, then $A_1 = 3A_2$, and if $t_1 = \frac{1}{2}t_2$, then $A_1 = \frac{1}{2}A_2$. In general, if $t_1 = kt_2$ for some positive constant k, then $A_1 = kA_2$. Now let t_1 and t_2 be any two time intervals and let $\frac{t_1}{t_2} = k$. So $t_1 = kt_2$, and, as just asserted, $A_1 = kA_2$. So $\frac{A_1}{t_1} = \frac{kA_2}{kt_2} = \frac{A_2}{t_2}$. Therefore, Kepler's law second can be reformulated as follows:

Let t be any time interval and let A_t be the area traced out by the planet during time t. Then the ratio $\frac{A_t}{t}$ is the same constant no matter what time t is taken and no matter where in the orbit the motion occurs.

Kepler published his third and final law in 1619.

3. Let M and J be any two planets and consider their elliptical orbits. Let a_M be the *semimajor axis* of the ellipse of M. This is $\frac{1}{2}$ the length of the major axis, i.e., the diameter through the two focal points. See Figure 4.4. Observe that the

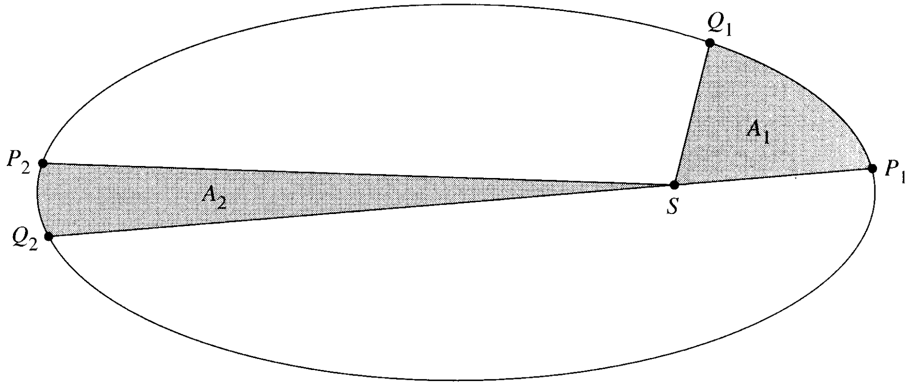


Figure 4.3

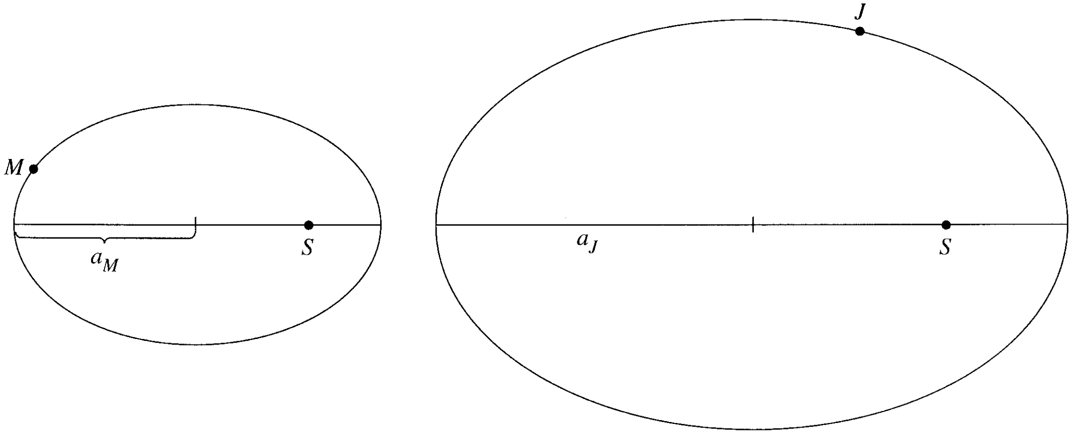


Figure 4.4

semimajor axis is a number and not a segment. Let T_M be the *period* of the orbit of planet M. This is the length of time required for one complete revolution. Let a_J and T_J be the semimajor axis and the period of planet J. Then Kepler's third law states that

$$\frac{a_M^3}{T_M^2} = \frac{a_J^3}{T_J^2}.$$

Notice that this law relates the sizes and periods of the various planetary orbits.

Kepler had hit upon the three laws that are fundamental to the explanation of the workings of the solar system. It should be pointed out that his scientific discourse also contains strange (by today's standards) theological and mystical elements.

Kepler's last years were troubled. His religious beliefs—which saw no contradiction between his astronomy and the Bible—were attacked, and he was labeled a heretic. Europe had become embroiled in the Thirty Years' War. This conflict between Catholic and Protestant forces for the control of Europe devastated the population and left many cities in ruins.

The new astronomy did not find general acceptance. Ptolemy's system of the universe was difficult to dislodge, because it rested on Aristotelian physics. This in itself was part of the comprehensive system of Aristotelian thought, which consists of a tight weave of philosophical discourse, explanations, and observations, deeply rooted and difficult to refute. It was not until Aristotelian learning was drawn into question that the new astronomy could make any serious headway. This task fell in large measure to Galileo Galilei.

4.2 The Studies of Galileo

Galileo Galilei (1564–1642) was born in Pisa. He exhibited an independence of outlook at an early age. He was sent to the University of Pisa, but he left four years later when his father could no longer afford the considerable fees. (Evidently, this was already a problem in those days.) A family friend who was a distinguished mathematician taught Galileo the elements of Euclid and Archimedes. After a time, Galileo mastered the basics of mathematics and physics. In time, however, he began to have questions. He devised experiments both in reality and in his mind to test what he was learning.

Aristotle's theory of motion proclaimed that the heavier a body is, the faster it falls. Galileo had problems with this, and he engaged in the following "thought" experiment. He contemplated a cannonball in free fall. Next, he considered it to be cut into two pieces, and he thought of both pieces falling side by side. See Figure 4.5. From the point of view of the fall, nothing had changed. Therefore, the cannonballs in the two situations must drop with the same speed. According to Aristotle, however, each of the two separate halves of the ball—being lighter—must drop more slowly than the ball as a whole. Aristotle had to be wrong. Galileo had discovered that all bodies fall in the same way, regardless of their weight, assuming that they are heavy enough for air resistance to be ignored. (Incidentally, most historians of science are of the opinion that Galileo did not drop cannonballs from the leaning tower of Pisa.)

Figure 4.5

Galileo next turned his attention to quantitative aspects of falling objects. Instead of considering a ball in free fall, he let it roll down an inclined plane. This slowed things down and made it easier to quantify observations. He fixed a unit of time and a unit of length. In an experiment undertaken in 1604, Galileo let a metal ball start from the top of the inclined plane at time $t = 0$. Careful measurements of times and distances (in repeated trials) revealed the following pattern. See Figure 4.6. The rolling ball covered a distance d between $t = 0$ and $t = 1$; a distance $3d$ between $t = 1$ and $t = 2$; a distance $5d$ between $t = 2$ and $t = 3$; a distance $7d$ between $t = 3$ and $t = 4$; and so on. Since average velocity is distance divided by time, Galileo observed that the successive average velocities of the ball were

$$\frac{d}{1}, \frac{3d}{1}, \frac{5d}{1}, \frac{7d}{1}, \frac{9d}{1}, \ldots,$$

and he noticed that the velocity increased at a constant rate. What he had discovered was this: the increase in the velocity of the ball divided by the time over which the increase occurs is the same constant no matter when the measurement is taken. This can be expressed algebraically as follows: Let $v(t)$ denote the velocity of the rolling ball at any time t. Let t_1 and t_2 with $t_1 < t_2$ be any two moments of time. Then

$$\frac{v(t_2) - v(t_1)}{t_2 - t_1} = C$$

for a constant C that is the same no matter what t_1 and t_2 are picked. The constant C does depend on the inclination of the plane, however.

Later, measuring time more accurately using water clocks (Galileo measured time in *tempi* with 1 *tempo* equal to 0.011 seconds) and linking free fall

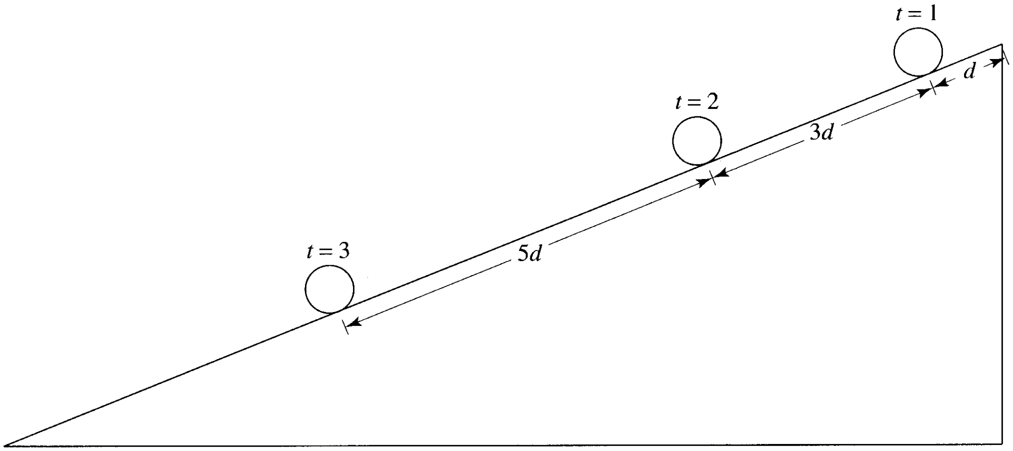


Figure 4.6

to properties of the motion of the pendulum, he discovered that in the case of a ball in free fall, the ratio

$$\frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

is also equal to a constant. Again, this ratio (nowadays written g) is the same regardless of the moments t_1 and t_2 at which the measurements are taken.

Beginning in 1608, Galileo put his inclined plane on a table and measured the horizontal and vertical distances between the point at which the ball left the edge of the table and the point at which it impacted the floor. After many trials with varying velocities, he found that the trajectory of the fall is a parabola. We will analyze one of these experiments[3] in detail in Chapter 9.

Aristotle's physics had also proclaimed that all objects have a natural motion toward the center of the universe, i.e., the center of the Earth, and that the motion of an object in any other direction is possible only as long as a propelling agent is in contact with it and drives it forward. Once this ceases to operate, the motion stops and the object falls straight to the ground or comes suddenly to rest. The propelling agent in the case of a projectile is the disturbance produced in the air at the time of the initial impulse. The air that is being pushed and compressed in front of the object rushes around behind it to prevent a vacuum (which cannot exist). In particular, for Aristotle, motion in a vacuum was inconceivable.

In summary, the fundamental principle of Aristotle's theory of motion is this: An object that is in motion will stop unless an external force continues to propel it.

Galileo, however, was skeptical and returned to his experiments. He released a ball at a certain elevation on an inclined plane. He let it roll up another inclined plane and observed that it reached its original height. See Figure 4.7. He took a second inclined plane with a smaller inclination and noticed that the same was true. Galileo continued this experiment in his mind, imagining frictionless inclined planes of smaller and smaller inclinations. Carrying his thinking to its logical end, he concluded that on a flat (and frictionless) surface, a perfectly spherical ball would roll forever.

Galileo had thus discovered the *law of inertia*: A body that is in a state of either rest or motion will continue in this same state, unless it is acted upon by some external force. This stands in sharp contrast to the hypothesis of Aristotle.

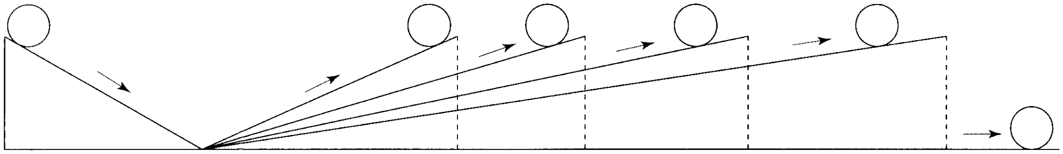


Figure 4.7

Aristotle's thinking does seem to be consistent with ordinary observation and the data supplied by common sense. (After all, a real spherical ball would not roll forever.) The laws of motion are not something likely to be discovered by experimental methods alone. At some point, a transposition in the mind of the scientist is also required. Galileo was aware of the "material impediments" of air resistance, friction, and the fact that the balls that rolled down his inclined planes were not perfectly round. He knew that these factors would lead to discrepancies between the conclusions of an experiment and the assertion of the basic laws of motion. But he knew also that these impediments must be separated from the fundamental principles. In ordinary experience, there are no perfectly spherical balls moving on perfectly smooth, frictionless inclined planes and rolling away to infinity. But by imagining them, Galileo stripped away the inessentials and revealed the laws at the core.

The problem of motion was a problem of fundamental nature, one that could not be solved by better observation within the framework of the older systems of ideas. A new frame of mind was required. In other words, to understand what is going on it is necessary to think not about real bodies as we actually observe them in the real world, but about idealized bodies moving in an idealized world, one without friction and resistance. It calls for the contemplation of geometrical shapes moving in abstract space. Upon arriving at the basic principles in this abstracted context, the variables of drag, friction, etc., can be reinserted and analyzed. Observe that Greek geometry had done a similar thing in a much simpler context. Its focus was not on the wheel and the ball, but on their abstractions, the circle and the sphere.

Galileo also had a considerable impact on astronomy. He was an advocate of the Copernican system. He heard about the invention of the telescope, fashioned his own, and turned it towards the sky with spectacular results. In January of 1610 he observed Jupiter. Diagrams in his book the *Starry Messenger* describe what he observed. See Figure 4.8. He discovered, sometimes on the left, sometimes on the right, and at times hidden from view altogether, four satellites in orbit around Jupiter. The existence of this system, which certainly does not have the Earth at its center, gave support to the viewpoint of Copernicus.

Galileo studied the phases of Venus and concluded that the pattern they exhibit is consistent with the heliocentric hypothesis. In the Ptolemaic system of

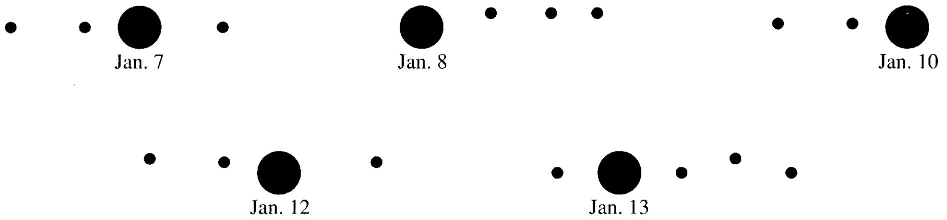


Figure 4.8

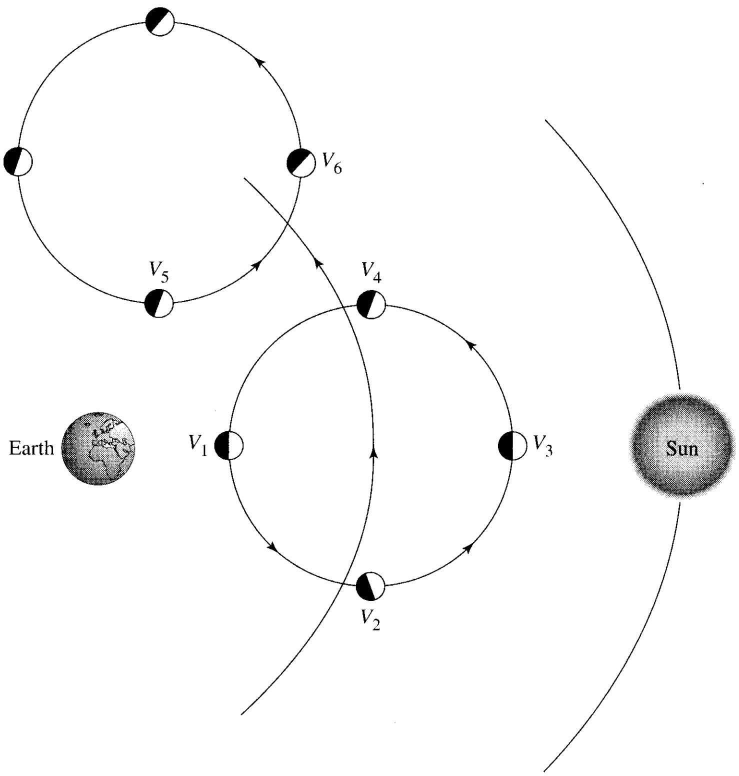


Figure 4.9

Figure 4.9, Venus (as seen from Earth) always appears more or less crescent-shaped. In the Copernican system of Figure 4.10, however, Venus has the full range of phases that Galileo observed. Therefore, the basic configuration of the Ptolemaic model had to be in error and that of Copernicus correct.

Galileo became professor of mathematics at the University of Padua. He was an eloquent speaker, an excellent teacher, and his fame soon spread. When a great hall with a capacity of more than a thousand proved too small, his astronomy lecture would be moved outside. He decided to publish his findings in pamphlets, dialogues, letters, and books. He wrote, not in the Latin of the academy, but in Italian, the language of his country. He became embroiled in a lengthy controversy with the Church. The Copernican system as a mathematical theory was one thing, but as a physical reality it was contradictory to scripture, undermined the authority of the church, and it had to be fought. After 20 years of hearings, misunderstandings, charges, revisions, and injunctions, Galileo was forced to recant and the Inquisition sentenced him to house arrest for life. After a lengthy illness, he died in 1642.

As we have seen, Galileo's contributions to science were twofold: His experiments established some of the basic laws of motion, and his discoveries with the telescope gave credibility to the Copernican heliocentric point of view. Perhaps even more important was his new approach to the understanding of natural phenomena, which facilitated the transition to modern science.

The definitive explanations of the discoveries of Galileo and Kepler require not only a new approach

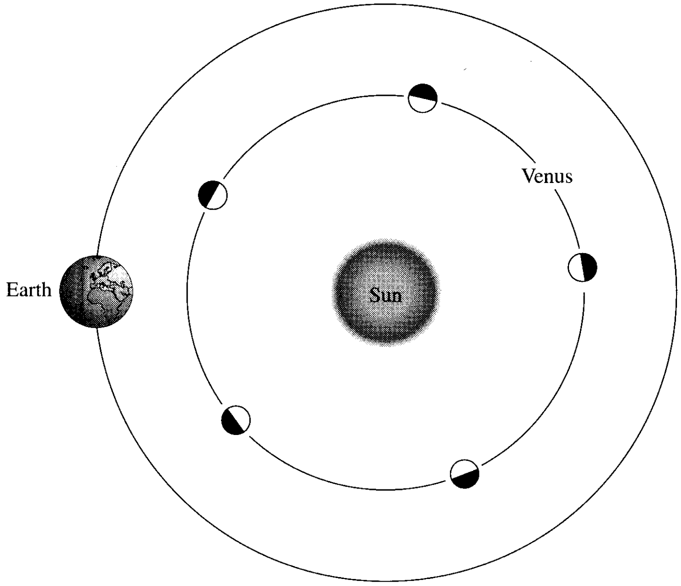


Figure 4.10

to science, but also a new mathematics. It is possible to extract considerable and essential quantitative information from Galileo's discovery that for falling bodies,

$$\frac{v(t_2) - v(t_1)}{t_2 - t_1} = \text{constant.}$$

The same is true with Kepler's laws. The fact that a planet's orbit is an ellipse with the Sun at one of the focal points, together with the equality

$$\frac{A_t}{t} = \text{constant}$$

contains precise quantitative information about the orbits of the planets. In both cases, however, the extraction of this information requires a new mathematics: differential and integral calculus.

The development of both of these branches of calculus required a new geometry. It is impossible to credit a single individual for the discovery of this "analytic geometry." Some of its basic principles have earlier origins. They can be seen, for example, in the work of Apollonius and in the coordinate system of Ptolemy's maps. Most historians agree, however, that the primary credit for analytic geometry as a modern subject goes to the French mathematician René Descartes (1596–1650).

Descartes received his early education from the Jesuits, who recognized both his physical weakness and his mental alertness. They allowed him to read and meditate in bed past rising hours. This became a lifelong habit that Descartes found very conducive to intellectual output. At the age of 22, amazingly, he became a professional soldier. The lulls in military activity during the winter months allowed him to pursue mathematics and philosophy. One day in 1619, while in service in Bavaria, he escaped the cold by shutting himself into a "stove." He recounted that, during his meditations in this especially warm room, a divine spirit revealed a new philosophy to him. When he emerged from the "stove," he had conceived the principles of analytic geometry and the idea of applying the mathematical method to philosophy.

His famous work *Discours de la méthode*, written in 1629 and published in 1637, recalls that

> Archimedes, in order that he might draw the terrestrial globe out of its place and transport it elsewhere [recall that Archimedes supposedly asserted, in reference to the lever, that if he had a place to stand he could move the world], demanded that only one point should be fixed and immovable; in the same way I shall have the right to conceive high hopes if I am happy enough to conceive one thing only which is certain and indisputable.

He then hit upon the most famous assertion in all of philosophy: Je pense, donc je suis. Cogito ergo sum. I think, therefore I am.

Our interest, however, is in a hundred-page appendix to the *Discours de la méthode*, namely *La Géométrie*. In it, Descartes experimented with algebraic notation: he used a; b, c to denote constant quantities; he wrote a^2, a^3, a^4, and $\sqrt{a^2+b^2}$ as we do today, and he wrote x, y, z for variable quantities. But more importantly in the current context, he fixed a point's position in the plane by assigning to it two "coordinates," namely, its distances to two perpendicular lines. He also discussed the relationship between graphs and equations that this connection provided. We now turn to a detailed description—in modern notation—of what Descartes achieved.

4.3 The Geometry of Descartes

Just as the points on a line can be identified with the real numbers by using a coordinatized line, so the points in a plane can be identified with pairs of real numbers. Start by drawing two perpendicular coordinatized lines that intersect at their origins. One line is placed horizontally with the positive direction to the right. The other is placed vertically with the positive direction upward. These are the *coordinate axes*. The horizontal axis is often called the *x-axis*, and the vertical axis the *y-axis*. However, these designations depend on the variables used. For example, in a context where a variable time t or angle θ is considered, an axis might be labeled by t or θ.

Any point P in the plane determines a unique ordered pair of numbers as follows. Refer to Figure 4.11. Draw the line through P parallel to the y-axis. This line intersects the x-axis at some point and this point has a coordinate, say a, on this number line. Similarly, draw a line through P parallel to the x-axis to get a coordinate, say b, on the y-axis. In this way, Descartes assigns to the point P the pair of numbers (a, b). The first number, a, is called the *x-coordinate* of P, and the second number, b, is called the *y-coordinate* of P. We say that P is the point with coordinates (a, b), and we will often write (a, b) in place of P. For example, the point S in Figure 4.11 has coordinates $(3, -3.2)$, so $S = (3, -3.2)$. By reversing the process we can start with an ordered pair, say (c, d), and arrive at the corresponding point R. For example, $(2, 4\frac{1}{2})$ corresponds to the point Q. The point $(0, 0)$ is called the *origin* and is often denoted by O. The notation 0 (zero) will be used for this point if the attention is on the x-axis or the y-axis.

This setup is called the *rectangular* or *Cartesian* (for Descartes) *coordinate system*. It provides the connection between geometric objects—points, collections of points, and curves—and algebra—pairs of numbers, collections of such pairs, and equations. The plane supplied with a coordinate system is called the *Cartesian plane*. The coordinate axes divide the Cartesian plane into four *quadrants*, labeled **I**, **II**, **III**, and **IV** in Figure 4.11. Notice that the first quadrant consists of the points whose x- and y-coordinates are both positive.

Check that the distance between the points -5 and 3 on the number line is 8. Observe that this is equal to the absolute value of $-5-3$ or, equivalently, that of $3-(-5)$. Similarly, the distance between 6.4 and 4.6 is 1.8. This is the absolute value of either $6.4-4.6$ or $4.6-6.4$. In general, the distance between the points a and b on a number line is equal to $a-b$ if $b \leq a$ and $b-a$ if $b > a$. See Figure 4.12. In the first case, $a-b \geq 0$ and hence $|a-b| = a-b$. In the second case, $a-b < 0$ and hence $|a-b| = -(a-b) = b-a$. Observe,

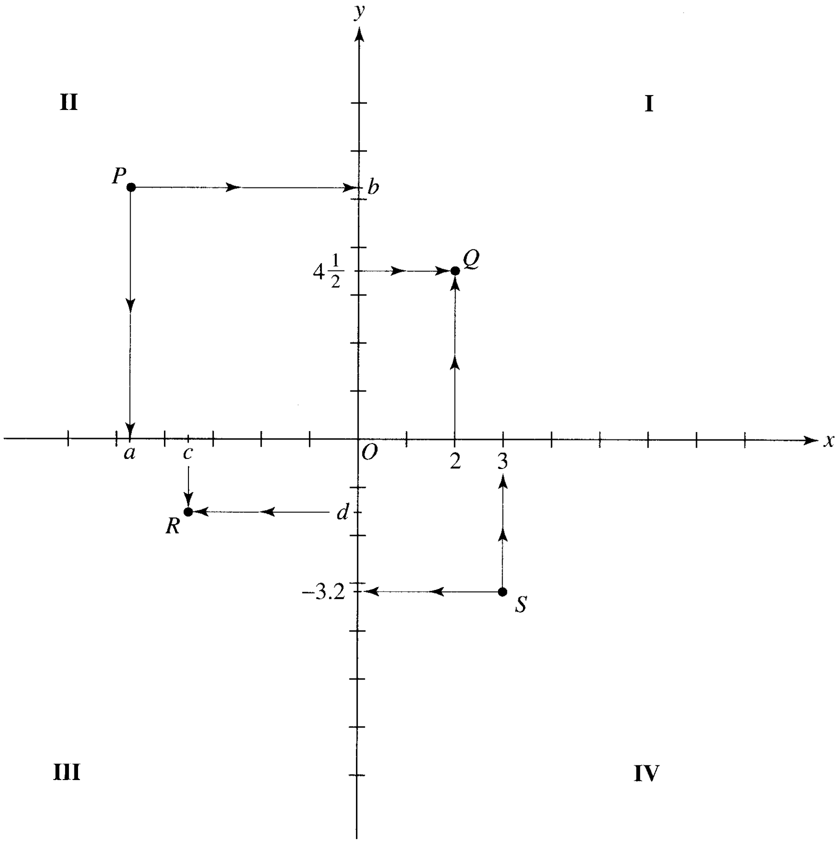


Figure 4.11

therefore, that the distance between a and b is equal to the absolute value $|a - b| = |b - a|$ in either case.

Consider any two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the plane and refer to Figure 4.13. By the remarks just made, the distance between the two x-coordinates is $|x_1 - x_2|$ and the distance between the two y-coordinates is $|y_1 - y_2|$. It follows that the segment P_1P_2 is the hypotenuse of the right triangle of Figure 4.14. Therefore, by the Pythagorean theorem,

$$P_1P_2 = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}.$$

Since $|x_1 - x_2|^2 = (x_1 - x_2)^2$ and $|y_1 - y_2|^2 = (y_1 - y_2)^2$, the distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is equal to

$$\boxed{P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}$$

Observe that we are continuing an earlier practice: When a segment, here P_1P_2, occurs in a mathe-

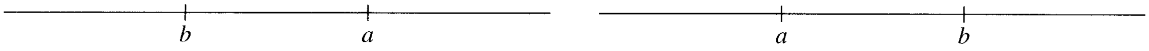


Figure 4.12

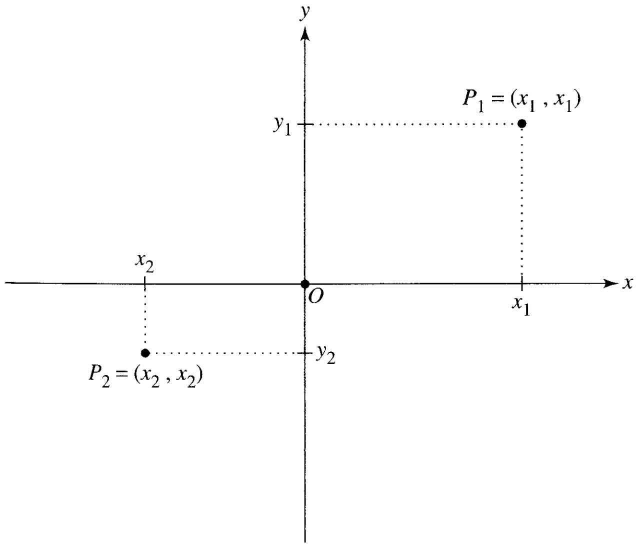


Figure 4.13

matical expression, the reference will be to the length of the segment.

Example 4.1. The distance between (1, –2) and (5,3) is

$$\sqrt{(1-5)^2 + (-2-3)^2} = \sqrt{4^2+5^2} = \sqrt{41}.$$

Since $\sqrt{(5-1)^2 + (3-(-2))^2}$ is also equal to $\sqrt{4^2+5^2} = \sqrt{41}$, the order in which the points are considered has no effect on the result.

The *graph* of an equation in x and y is the set of all points (a, b) in the plane with the property that the values $x = a$ and $y = b$ satisfy the equation. For example, the point $(1, -3)$ is a point on the graph of the equation $3x^2 + y^2 = 12$, since $3(1)^2 + (-3)^2 = 12$. The graph of the equation $x = 5$ is the collection of all points (x, y) with $x = 5$. This is the vertical line through the point $x = 5$ on the x-axis. Similarly, the graph of $y = -7$ is the horizontal line through the point $y = -7$ on the y-axis.

The *y-intercepts* of a graph are the y-coordinates of the points of intersection of the graph with the y-axis. They are found by setting $x = 0$ and solving for y. When this is done for the graph of $xy^3 + y^2 - x^3 + 3 = 0$, for example, we get $y^2 = 3$, and therefore $y = \pm\sqrt{3}$. In the same way, the *x-intercepts* of a graph are the x-coordinates of the points of intersection of the graph with the x-axis. They are found by setting $y = 0$ and solving for x. For $xy^3 + y^2 - x^3 + 3 = 0$, we get $x^3 = 3$; so $x = \sqrt[3]{3}$.

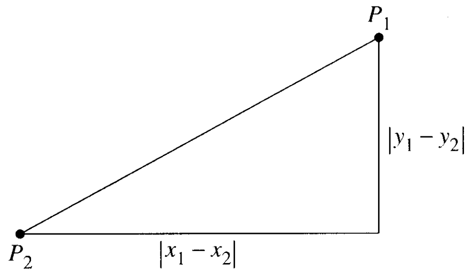


Figure 4.14

A graph is a pictorial representation of an equation. It provides a picture of the algebra and clarifies what is going on. In the other direction, when a curve has to be understood, it is often of great advantage to have an equation of the curve, that is to say, an equation whose graph is the curve in question. This translates the geometric concern into algebra. The algebra allows for precise computations that can reveal exact numerical information. In other words, if a curve can be represented by an algebraic equation, then the rules of algebra can be used to analyze it. This complementary duality between geometry on the one hand and numbers and algebra on the other is the essence of analytic geometry. This interplay is of crucial importance in the solution of many mathematical problems.

The parabola provides an example that illustrates the point. Take any parabola and move it so that its directrix D is horizontal. Refer to Figure 4.15. Suppose D crosses the y-axis at the point c and observe that $y = c$ is the equation of D. Let $F = (a, b)$ be the focus. Since F is not on D, it follows that $b \neq c$.

Let $P = (x, y)$ be any point in the plane. Then the distance from P to D is $|y - c|$ and the distance from

P to F is

$$PF = \sqrt{(x-a)^2 + (y-b)^2}.$$

So P is on the parabola precisely when $|y-c| = \sqrt{(x-a)^2 + (y-b)^2}$. Squaring both sides gives $(y-c)^2 = (x-a)^2 + (y-b)^2$. After multiplying things out and simplifying, we have

$$\begin{aligned} y^2 - 2cy + c^2 &= x^2 - 2ax + a^2 + y^2 - 2by + b^2 \\ 2by - 2cy &= x^2 - 2ax + a^2 + b^2 - c^2 \\ 2(b-c)y &= x^2 - 2ax + (a^2 + b^2 - c^2). \end{aligned}$$

Since $b \neq c$, $b - c \neq 0$. So we can divide through by $2(b-c)$ to get

$$(*) \quad y = \left(\frac{1}{2(b-c)}\right)x^2 - \left(\frac{a}{b-c}\right)x + \left(\frac{a^2+b^2-c^2}{2(b-c)}\right).$$

Taking $A = \frac{1}{2(b-c)}$, $B = -\frac{a}{b-c}$, and $C = \frac{a^2+b^2-c^2}{2(b-c)}$, we see that any parabola with a horizontal directrix has an equation of the form

$$y = Ax^2 + Bx + C.$$

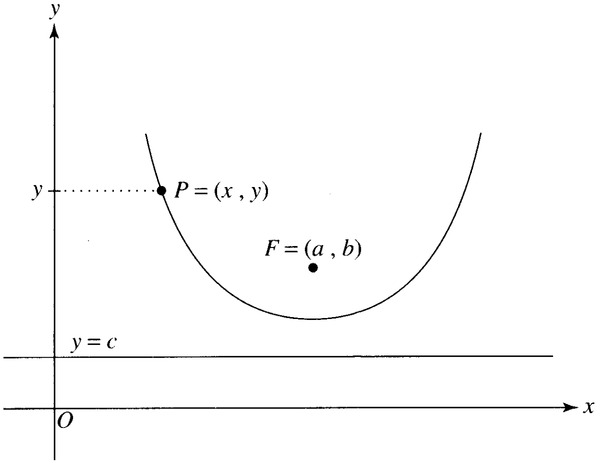


Figure 4.15

Example 4.2. Find an equation of the parabola with directrix the line $y = 5$ and focus $(2, -3)$.

Solution. Here $c = 5$, $a = 2$, and $b = -3$. Since

$$\frac{1}{2(b-c)} = \frac{1}{2(-3-5)} = -\frac{1}{16},$$

$$-\frac{a}{b-c} = -\frac{2}{-3-5} = \frac{1}{4}, \quad \text{and}$$

$$\frac{a^2+b^2-c^2}{2(b-c)} = \frac{4+9-25}{-8} = \frac{3}{4},$$

we see that $y = -\frac{1}{16}x^2 + \frac{1}{4}x + \frac{3}{4}$ is an equation of this parabola.

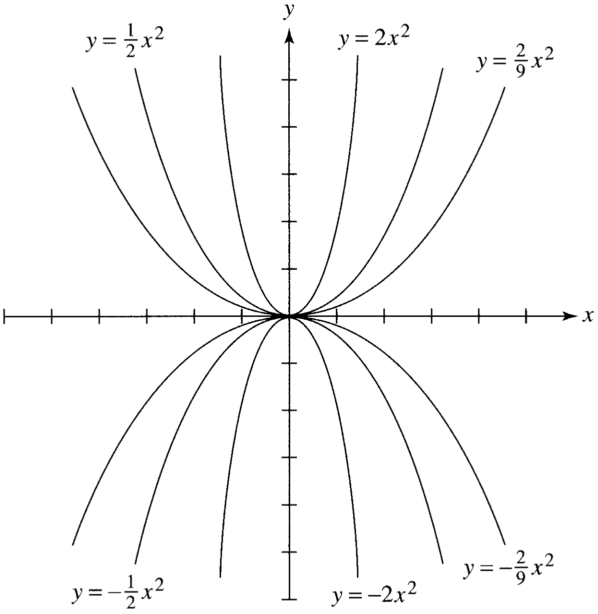


Figure 4.16

We have just established the fact that any parabola with a horizontal directrix has an equation of the form $y = Ax^2 + Bx + C$. This argument can be reversed to show that the graph of any equation of the form $y = Ax^2 + Bx + C$ (with $A \neq 0$) is a parabola with horizontal directrix.

Example 4.3. Show that the graph of the equation $y = 2x^2 + 5x - 4$ is a parabola and find its directrix and focus.

Solution. Work backwards using equation $(*)$. Since $\frac{1}{2(b-c)} = 2$, it follows that $b - c = \frac{1}{4}$. Because $-\frac{a}{b-c} = 5$, we now get that $a = -\frac{5}{4}$. From the equality $-4 = \frac{a^2+b^2-c^2}{2(b-c)}$ we get $a^2 + b^2 - c^2 = -4 \cdot 2(b - c)$. So $b^2 - c^2 = -a^2 - 4 \cdot 2(b - c) = -\frac{25}{16} - \frac{8}{4} = -\frac{57}{16}$. Use of the factorization $b^2 - c^2 = (b + c)(b - c)$ shows that $b + c = -\frac{57}{16} \cdot 4 = -\frac{57}{4}$. Since $b - c = \frac{1}{4}$, it is now easy to solve for b and c to get $b = -7$ and $c = -7\frac{1}{4}$. It follows from equation $(*)$ that $y = 2x^2 + 5x - 4$ is the equation of the parabola with directrix $y = -7\frac{1}{4}$ and focus $\left(-\frac{5}{4}, 7\right)$.

Consider the equation $y = Ax^2 + Bx + C$, with $A \neq 0$. For very large positive or negative x the term Ax^2 will dominate the others. So if $A > 0$, then y is positive when $|x|$ is large, and if $A < 0$, then y is negative when $|x|$ is large. It follows that the parabola $y = Ax^2 + Bx + C$ opens upward if $A > 0$ and downward if $A < 0$.

Example 4.4. Draw the graph of the parabola $y = 2x^2$.

Solution. Set up a table of values, plot points, and join them by a smooth curve to obtain the appropriate graph in Figure 4.16, which also shows two other parabolas of the form $y = Ax^2$.

Example 4.5. The parabola $y = x^2 + 3x + 4$ opens upward. For which value of x does y attain its smallest value?

Solution. Complete the square to rewrite the equation as

$$y = x^2 + 3x + 4 = x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4$$

$$= \left(x + \frac{3}{2}\right)^2 + 4 - \frac{9}{4}.$$

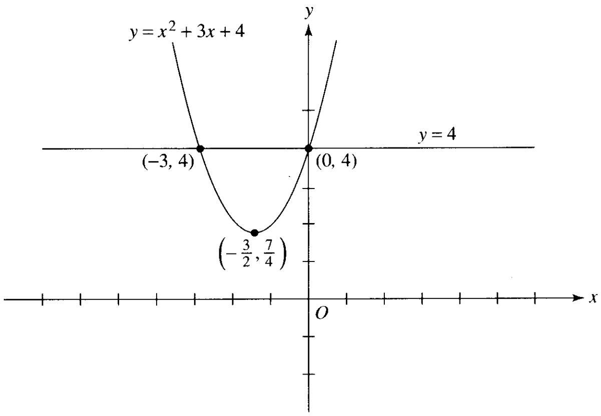


Figure 4.17

So $y = \left(x + \frac{3}{2}\right)^2 + \frac{7}{4}$. Notice that the smallest possible value of y is $y = \frac{7}{4}$ and it occurs for $x = -\frac{3}{2}$.

Example 4.6. Cut the parabola $y = x^2 + 3x + 4$ with the horizontal line $y = 4$ and compute the area of the resulting parabolic section. See Figure 4.17.

Solution. This can be done using Archimedes's theorem. By Example 4.5 $\left(-\frac{3}{2}, \frac{7}{4}\right)$ is the lowest point on the parabola. Since the tangent line hugs the curve near the point of contact, the tangent to the parabola at $\left(-\frac{3}{2}, \frac{7}{4}\right)$ must be horizontal. So this tangent is parallel to the cut $y = 4$. It follows that $\left(-\frac{3}{2}, \frac{7}{4}\right)$ is the vertex of the parabolic section. (See Section 3.1.) The points of intersection of the parabola and the line $y = 4$ are needed next. If (x, y) is on both the parabola and $y = 4$, then $y = x^2 + 3x + 4$ and $y = 4$. So $x^2 + 3x + 4 = 4$, and $x^2 + 3x = 0$. By factoring, $x^2 + 3x = x(x + 3) = 0$, so $x = 0$ or $x = -3$. Therefore the points of intersection are $(-3, 4)$ and $(0, 4)$. By Archimedes's theorem, the area of the parabolic section is $\frac{4}{3}$ times that of the inscribed triangle. Refere to Figure 4.17 again. The inscribed triangle has base 3 and height $4 - \frac{7}{4} = \frac{9}{4}$; so the triangle has area $\frac{1}{2} \cdot 3 \cdot \frac{9}{4} = \frac{27}{8}$. Thus the area of the parabolic section is $\frac{4}{3} \cdot \frac{27}{8} = \frac{9}{2}$.

Consider the general equation $y = Ax^2 + Bx + C$ of the parabola. If the variables x and y are interchanged, then the roles of the x-axis and the y-axis are interchanged. It follows that any parabola with vertical directrix has an equation of the form $x = Ay^2+By+C$ with $A \neq 0$. It opens to the right if $A > 0$ and to the left if $A < 0$.

In the same way that curves in the Cartesian plane are represented by algebraic equations, regions in the plane are given by algebraic expressions involving inequalities.

Example 4.7. Describe and sketch the regions given by the following sets of points in the Cartesian plane.

(a) The set $\{(x, y) \mid x \geq 1\}$. The notation means that this is "the set of all points (x, y) with the property that $x \geq 1$."

(b) The set $\{(x, y) \mid |x| < 1 \text{ and } |y| < 1\}$. This is "the set of all points (x, y) with the property that $|x| < 1$ and $|y| < 1$."

Solution. (a) All points on or to the right of the line $x = 1$. (b) All points inside the box formed by the lines $x = 1$, $x = -1$, $y = 1$, and $y = -1$.

Example 4.8. Let $P = (x, y)$ be a point in the plane. What relationship between x and y places P into the parabolic section of Example 4.6?

Solution. The parabola has equation $y = x^2+3x+4$. Refer back to Figure 4.17 and start with any point (x, y) on the parabola. So $y = x^2+3x+4$. Increasing the y-coordinate moves the point above the parabola. It follows that the points (x, y) on or above the parabola are precisely those with $y \geq x^2 + 3x + 4$. Therefore, the points $P = (x, y)$ in question are those that satisfy $4 \geq y \geq x^2 + 3x + 4$. So the parabolic section is $\{(x, y) \mid x^2 + 3x + 4 \leq y \leq 4\}$.

4.4 Circles and Trigonometry

Consider a circle with center $C = (3, 2)$ and radius 4. By definition, a point $P = (x, y)$ in the plane is on the circle precisely if the distance from P to the center $(3, 2)$ is equal to 4. See Figure 4.18. Put another way, $P = (x, y)$ is on the circle exactly when

$$PC = \sqrt{(x-3)^2 + (y-2)^2} = 4.$$

Squaring both sides, we see that this is so precisely when

$$(x-3)^2 + (y-2)^2 = 16.$$

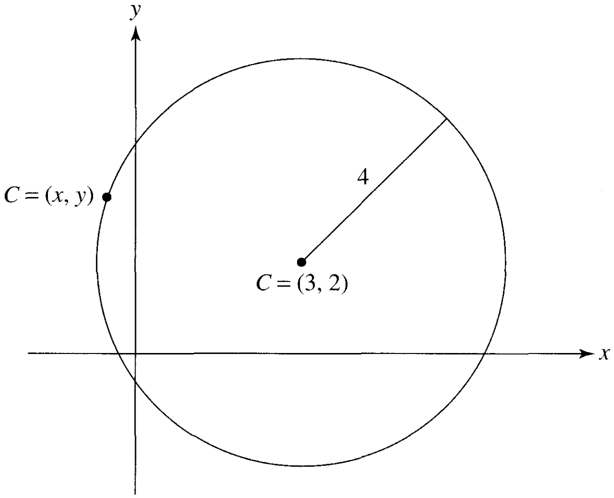


Figure 4.18

Proceeding in the same way, a point $P = (x, y)$ is on the circle with radius r and center (h, k) exactly when its coordinates satisfy

$$(x-h)^2 + (y-k)^2 = r^2.$$

This is an equation of the circle with radius r and center (h, k). If the center is the origin $O = (0, 0)$, the equation is $x^2 + y^2 = r^2$.

Example 4.9. Find an equation of the circle with radius 3 and center $(2, -5)$.

Solution. Putting $r = 3$, $h = 2$, and $k = -5$, we obtain $(x-2)^2 + (y+5)^2 = 9$.

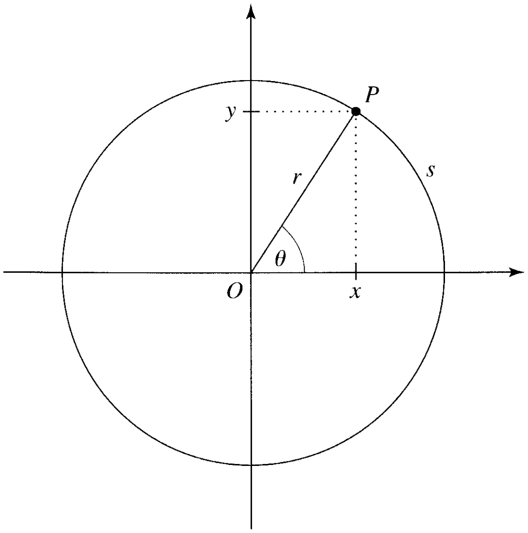


Figure 4.19

Example 4.10. Show that the graph of the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ is a circle and find its center and radius.

Solution. Rewrite the equation as

$$(x^2 + 2x) + (y^2 - 6y) = -7.$$

After completing the square for each variable and adding the appropriate constants to the right side of the equation, this equation becomes

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9,$$

or

$$(x + 1)^2 + (y - 3)^2 = 3.$$

Comparing this equation with the standard equation of the circle, we see that $h = -1$, $k = 3$, and $r = \sqrt{3}$. So the given equation is that of a circle with center $(-1, 3)$ and radius $\sqrt{3}$.

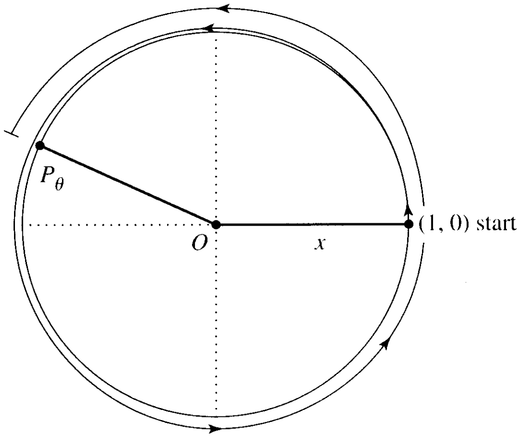


Figure 4.20

Consider a circle of radius r with center the origin. Let $P = (x, y)$ be a point on the circle in the first quadrant. Draw a radius to P and let θ be its angle with the positive x-axis. Denote by s the length of the arc that this angle cuts from the circumference; see Figure 4.19. Observe that $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, and that $\theta = \frac{s}{r}$ (in radians). So

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

When $r = 1$, these relationships have the simpler form

$$x = \cos\theta, \quad y = \sin\theta, \quad \text{and} \quad \theta = s.$$

We will therefore take $r = 1$. This circle with center at the origin and radius 1 is called the *unit circle*. Its equation is $x^2 + y^2 = 1$ and its circumference is 2π.

So far—see Chapter 1—only angles θ with radian measure $0 \le \theta \le 2\pi$ have been considered. In fact, in the definitions of $\sin\theta$ and $\cos\theta$ it was required that $0 \le \theta \le \frac{\pi}{2}$. We will now expand the concepts of angle, sine, and cosine. After our discussion is complete, we will have achieved the following: Any real number θ can be interpreted as an angle, and $\sin\theta$ and $\cos\theta$ will make sense for any real number θ.

Let θ be any real number. Assume first that $\theta \ge 0$. Measure off a distance of θ along the perimeter of the unit circle: start at the point (1, 0) and proceed in a *counterclockwise* direction. Think of winding a string of length θ around a cylinder of radius 1. It may be necessary to go around the circle many times. After the entire distance θ is rolled out, place a point on the circle to mark the measurement and label the point P_θ. Recall that the circumference of the unit circle is

2π; so $\frac{\pi}{2}$ is the length of a quarter of the circle. To measure off $\theta = \frac{\pi}{2}$, start at (1, 0), go around (always counterclockwise) the first quarter of the circle, and stop at $P_\theta = (0,1)$. To measure off $\theta = \pi = 2 \cdot \frac{\pi}{2}$, go around the first two quarters of the circle and stop at $P_\theta = (-1,0)$. For $\theta = 1\frac{1}{2}\pi = 3 \cdot \frac{\pi}{2}$, go around the first three quarters to $P_\theta = (0,-1)$. For $\theta = 2\pi = 4 \cdot \frac{\pi}{2}$, go around all four quarters to $P_\theta = (1,0)$. Note that this was the starting point. For $\theta = 2\frac{1}{2}\pi = 5 \cdot \frac{\pi}{2}$ continue for one more quarter, for a total of five quarters, to $P_\theta = (0,1)$. For $\theta = 5\pi = 10 \cdot \frac{\pi}{2}$, go around ten quarters. After four quarters, and again after eight quarters, you'll reach the starting point (1, 0), then you'll stop at $P_\theta = (-1,0)$. Consider $\theta = 8.691$. Since $5 \cdot \frac{\pi}{2} \approx 7.854 \le 8.691 \le 9.425 \approx 6 \cdot \frac{\pi}{2}$, go around the perimeter for five quarters to (0, 1), and then for another $8.691 - \frac{5}{2}\pi \approx 0.837$ units past this point to get to P_θ. See Figure 4.20. It should now be clear how any positive number θ can be interpreted as an angle: it is the opening generated by the segment from O to (1, 0) as it rotates from (1, 0) to the point P_θ.

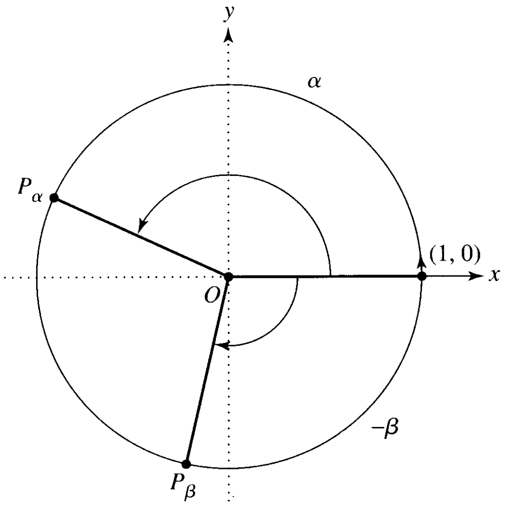


Figure 4.21

If θ is negative, then $-\theta$ is positive. To get the point P_θ in this case, measure off $-\theta$ on the circumference in a *clockwise* direction, again starting at (1, 0). Therefore, when the segment from O to (1, 0) rotates counterclockwise, it generates *positive* angles and when it rotates in a clockwise fashion, it generates *negative* angles. In either case, we will refer to the real number θ as the radian measure of the angle obtained in this way. Convince yourself that for $0 \le \theta \le 2\pi$ this use of the concept radian measure is the same as that already introduced in Section 1.2. Figure 4.21 gives an example of a positive angle α and a negative angle β. What is the radian measure of the angle obtained by letting the segment from O to (1, 0) rotate clockwise for three and a quarter revolutions? Since three and a quarter revolutions are $3\frac{1}{4} \times 4 = 13$ quarter turns, this angle is $-\frac{13}{2}\pi$. All angles can be given in degrees also. For example, the angle just discussed is equal to $\left(-3\frac{1}{4}\right) \cdot 360^\circ = -1170^\circ$. However, in this book, angles will virtually always be given in radians.

We have defined angles in terms of the length of the arc that they cut from the unit circle. A circle of any radius r can also be used for this purpose, but then division by r is necessary.

We now turn to the definition of $\sin\theta$ and $\cos\theta$ for any real number θ. Continue to consider the unit circle. Take θ and locate the point P_θ. Let the coordinates of P_θ be x and y and define

$$\cos\theta = x \quad \text{and} \quad y = \sin\theta$$

as illustrated in Figure 4.22. To repeat, $\sin\theta$ and $\cos\theta$ are defined to be the y-coordinate and the x-coordinate of the point P_θ, respectively. For $0 \le \theta \le \frac{\pi}{2}$, this agrees with the definition in Section 1.4. Refer to Figure 1.16 in that section and take $h = 1$.

Suppose now that we are given a circle of radius r and a point Q on the circle and that θ is the angle between the radius OQ and the positive x-axis. What are the coordinates of Q? Consider the corresponding point $P_\theta = (\cos\theta, \sin\theta)$ on the unit circle. See Figure 4.23. By similar triangles,

$$\frac{x}{\cos\theta} = \frac{y}{\sin\theta} = \frac{r}{1} = r.$$

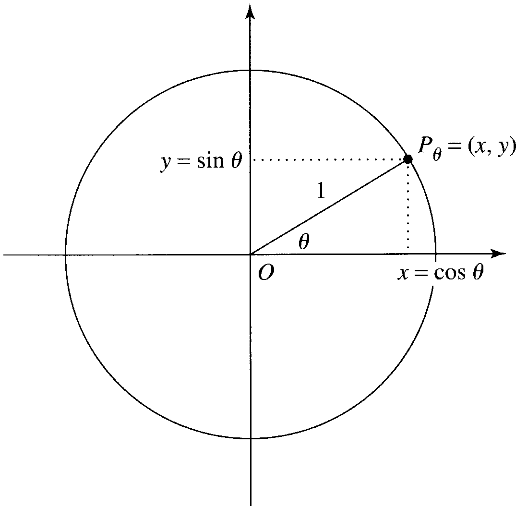


Figure 4.22

Therefore, the coordinates of Q are

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

The sine and cosine satisfy a number of basic formulas. These can often be "read off" from Figure 4.22. Let θ be any real number. Since the point $P_\theta = (\cos\theta, \sin\theta)$ is on the unit circle, it satisfies the equation $x^2 + y^2 = 1$. Therefore,

$$\sin^2\theta + \cos^2\theta = 1.$$

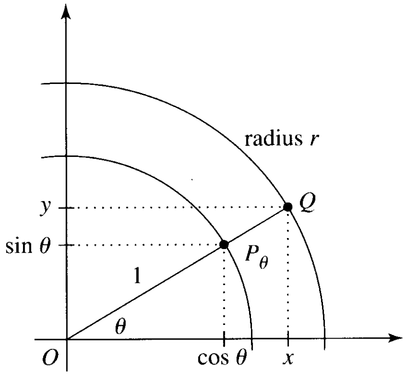


Figure 4.23

Example 4.11. Let θ be any real number. Observe that P_θ and $P_{\theta+2\pi}$ are the same point. So

$$\sin(\theta + 2\pi) = \sin\theta \quad \text{and} \quad \cos(\theta + 2\pi) = \cos\theta.$$

Why are these formulas valid with -2π in place of 2π? Note that the points P_θ and $P_{\theta+\pi}$ are on the opposite ends of a diagonal of the unit circle. After thinking for a moment (and perhaps making a sketch of the situation), you will see that

$$\sin(\theta + \pi) = -\sin\theta \quad \text{and} \quad \cos(\theta + \pi) = -\cos\theta.$$

Why are these formulas valid with $-\pi$ in place of π? Think about how the points P_θ and $P_{-\theta}$ are related and observe that $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$.

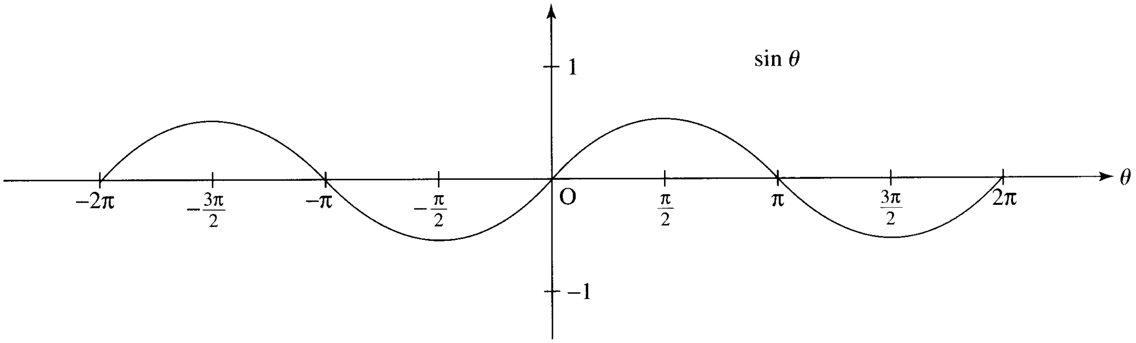


Figure 4.24

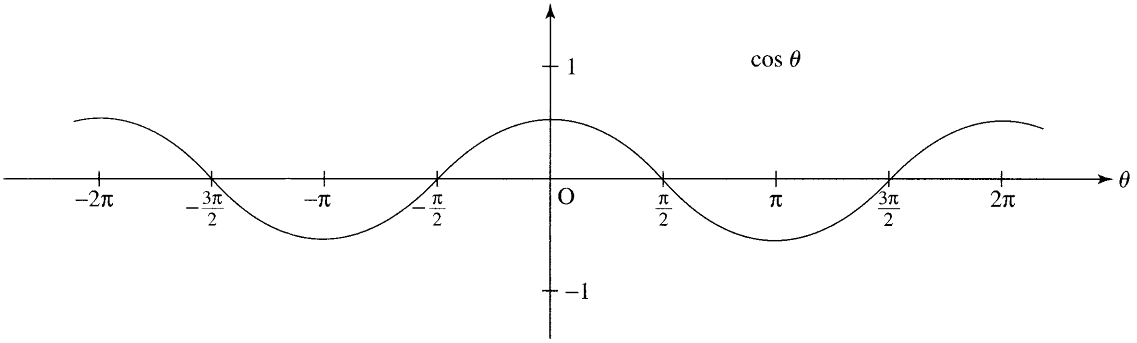


Figure 4.25

Continue to refer to Figure 4.22. Notice that as θ varies from 0 to $\frac{\pi}{2}$, $\sin\theta$ varies from 0 to 1; as θ moves from $\frac{\pi}{2}$ to π, $\sin\theta$ varies from 1 to 0; and so on. Plotting the various points $(\theta, \sin\theta)$ provides the graph in Figure 4.24. Doing the same thing for the cosine, we get the graph in Figure 4.25.

Now turn to $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Suppose first that $0 \le \theta \le \frac{\pi}{2}$. If $\theta = 0$, then tan $= 0$. Again turn to Figure 4.22. As θ increases, the sine increases and the cosine decreases. Since both are positive, $\tan\theta$ increases. When θ is close to $\frac{\pi}{2}$, the sine is close to 1 and the cosine is close to 0, so $\tan\theta$ is very large. If $\theta = \frac{\pi}{2}$, then $\cos\theta = 0$, so that $\tan\theta$ is not defined. When $-\frac{\pi}{2} \le \theta \le 0$, the situation is similar. Since the sine is negative and the cosine is positive, $\tan\theta$ is now negative. The rest of the graph of the tangent is simply a repitition of the pattern for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. It is sketched in Figure 4.26.

Example 4.12. Let θ be any angle. Since $\sin(\theta + \pi) = -\sin\theta$ and $\cos(\theta + \pi) = -\cos\theta$, it follows that $\tan(\theta + \pi) = \tan\theta$. Similarly, $\tan(-\theta) = -\tan\theta$.

4.5 The Ellipse

Consider an ellipse given by the focal points F_1 and F_2 and the constant $k \ge F_1F_2$. Recall from

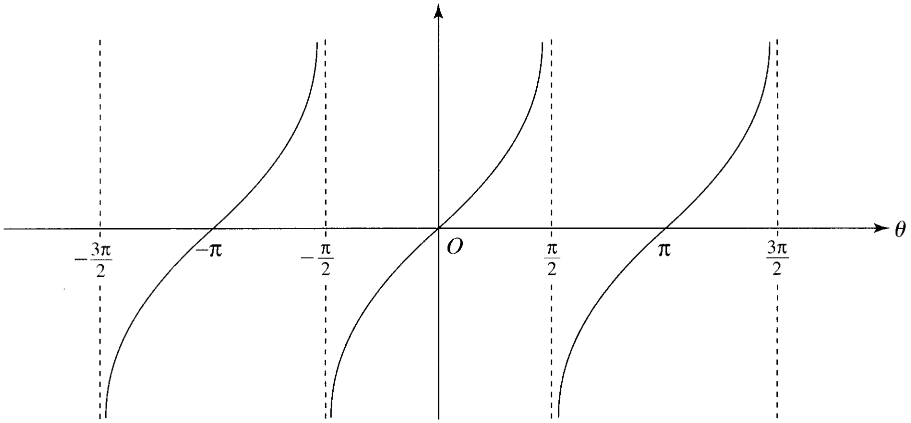


Figure 4.26

Section 3.1 that k is the sum of the distance from a point on the ellipse to F_1 and the distance from that point to F_2. Move the ellipse in the plane in such a way that F_1 and F_2 lie on the x-axis equidistant from the origin. Refer to Figure 4.27. Then the origin is the center of the ellipse. The ellipse is now in what we call *standard position.* Note that in this position, there is a number $e \geq 0$ such that

$$F_1 = (e, 0) \quad \text{and} \quad F_2 = (-e, 0)$$

and $k \geq 2e$. We already know from Section 3.1 that a point $P = (x, y)$ falls on the ellipse exactly when $PF_1 + PF_2 = k$. The concern will now be the precise conditions on the coordinates x and y of P that will guarantee this. Let $a = \frac{k}{2}$. So $2a = k \geq 2e$, and hence $a \geq e$. If $a = e$, then $k = 2e$. In this case, the ellipse consists of the points on the segment F_1F_2. (Why?)

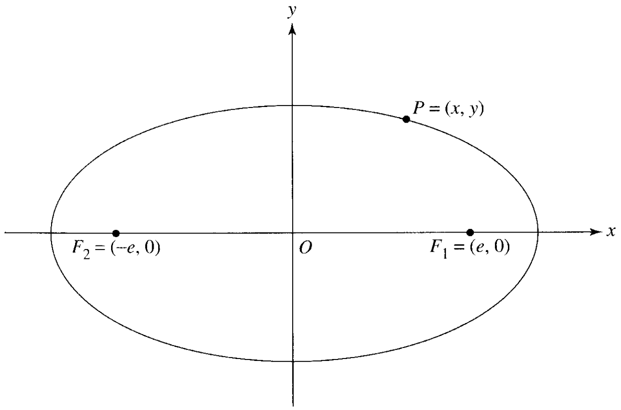


Figure 4.27

Since the ellipse consists of all points $P = (x, y)$ such that $PF_1 + PF_2 = k$, the point $P = (x, y)$ is on the ellipse precisely if

$$\sqrt{(x-e)^2 + (y-0)^2} + \sqrt{(x-(-e))^2 + (y-0)^2} = 2a.$$

After simplifying, squaring both sides, canceling, moving things around, and simplifying once more, this equation is transformed in successive steps to

$$\sqrt{(x-e)^2 + y^2} + \sqrt{(x+e)^2 + y^2} = 2a,$$

$$\sqrt{(x-e)^2 + y^2} = 2a - \sqrt{(x+e)^2 + y^2},$$

$$(x-e)^2 + y^2 = 4a^2 - 4a\sqrt{(x+e)^2 + y^2} + (x+e)^2 + y^2,$$

$$(x-e)^2 = 4a^2 - 4a\sqrt{(x+e)^2 + y^2} + (x+e)^2,$$

$$x^2 - 2ex + e^2 = 4a^2 - 4a\sqrt{(x+e)^2 + y^2} + x^2 + 2ex + e^2,$$

$$a\sqrt{(x+e)^2 + y^2} = a^2 + ex,$$

$$a^2\left((x+e)^2 + y^2\right) = a^4 + 2a^2ex + e^2x^2,$$

$$a^2(x^2 + 2ex + e^2 + y^2) = a^4 + 2a^2ex + e^2x^2,$$

$$a^2x^2 + 2a^2ex + a^2e^2 + a^2y^2 = a^4 + 2a^2ex + e^2x^2,$$

$$a^2x^2 - e^2x^2 + a^2y^2 = a^4 - a^2e^2,$$

$$(a^2 - e^2)x^2 + a^2y^2 = a^2(a^2 - e^2).$$

Now set $b = \sqrt{a^2 - e^2}$. Since $b^2 = a^2 - e^2$, this last equation becomes $b^2x^2 + a^2y^2 = a^2b^2$. In the case $a = e$, it was already pointed out that the ellipse consists of the points on the segment F_1F_2. So we will now assume that $a > e$ and hence that $b > 0$. Dividing the equation $b^2x^2 + a^2y^2 = a^2b^2$ through by a^2b^2 now shows that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is the *standard equation* of the ellipse. Since $b = \sqrt{a^2 - e^2}$ notice that $a \geq b$. Observe that if $a = b$, then this ellipse is a circle with radius a.

Setting $y = 0$ gives $x^2 = a^2$. So the x-intercepts are $\pm a$. It follows that $k = 2a$ is the length of the major axis. Therefore a is equal to the *semimajor* axis. Similarly, the y-intercepts are $\pm b$, the length of the minor axis is $2b$, and b is called the *semiminor* axis.

The distance e between the center and either of the focal points is called the *linear eccentricity* of the ellipse. Refer to Figure 4.28. Since the point $B = (0, b)$ is on the ellipse, $2BF_1 = BF_1 + BF_2 = k = 2a$, and hence $BF_1 = a$. In the same way, $BF_2 = a$. By Pythagoras's theorem, $a^2 = b^2 + e^2$. Hence, $e^2 = a^2 - b^2$. So the linear eccentricity is equal to $e = \sqrt{a^2 - b^2}$. The ratio

$$\varepsilon = \frac{\text{linear eccentricity}}{\text{semimajor axis}} = \frac{e}{a}$$

is the *astronomical eccentricity*. Since $e \leq a$, it is clear that $0 \leq \varepsilon \leq 1$. The constant ε measures the flatness of the ellipse. If $\varepsilon = 0$, then $e = 0$, in which case $F_1 = F_2$ and the ellipse is a circle. If $\varepsilon = 1$ then $e = a$. This is the case where the ellipse is the line segment between the two focal points. The "general ellipse" has a shape between these two extremes. If it is close to being a circle, then b is close to a, the eccentricity e is small, and ε is close to zero. If the ellipse is flat, then b is small relative to a, the quantities e and a are close to each other, and ε is close to 1.

Example 4.13. Show that the graph of $9x^2 + 16y^2 = 144$ is an ellipse. Determine the semimajor and semiminor axes, the two eccentricities, the two focal points, and the constant k.

Solution. Divide both sides of the equation by 144 to get

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

This is the standard equation of an ellipse. Since $a^2 = 16$ and $b^2 = 9$, we get $a = 4$ and $b = 3$. The linear and astronomical eccentricities are, respectively, $e = \sqrt{16 - 9} = \sqrt{7}$ and $\varepsilon = \frac{\sqrt{7}}{4} = 0.66$. The focal points are $(-\sqrt{7}, 0)$ and $(\sqrt{7}, 0)$, and $k = 2a = 8$.

With the ellipse placed with the focal points on the x-axis and the origin at the center of the segment connecting them, its equation—as we have seen—is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with $a \geq b$. If we line up the ellipse so that the focal points are on the y-axis (again with the center of the connecting segment at the origin), then the ellipse has an equation of exactly the same form, but this time with $b \geq a$.

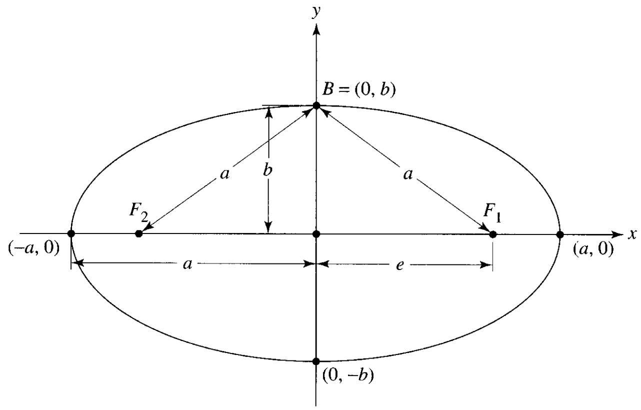


Figure 4.28

4.6 Cavalieri's Principle

We begin with an important general principle. Kepler made use of it, but it is known today as Cavalieri's principle. Bonaventura Cavalieri (1598–1644) was a student of Galileo.

Consider any two regions with areas C and D. Place both of them above the x-axis as in Figure 4.29. Suppose that for every x, the vertical cross-sectional cut through D has the same length as the vertical cross-sectional cut through C, that is $d_x = c_x$ for all x. In such a situation it is certainly reasonable to conclude that $C = D$. To reach this conclusion regard the two regions as sliced into thin strips. Therefore the underlying principle is the same as that of the *Method* of Archimedes. It is equally reasonable to conclude: If $d_x = 2c_x$ for all x, then $D = 2C$; if $d_x = 3c_x$ for all x, then $D = 3C$; if $d_x = \frac{1}{2}c_x$ for all x, then $D = \frac{1}{2}C$, and so on. We have now arrived at

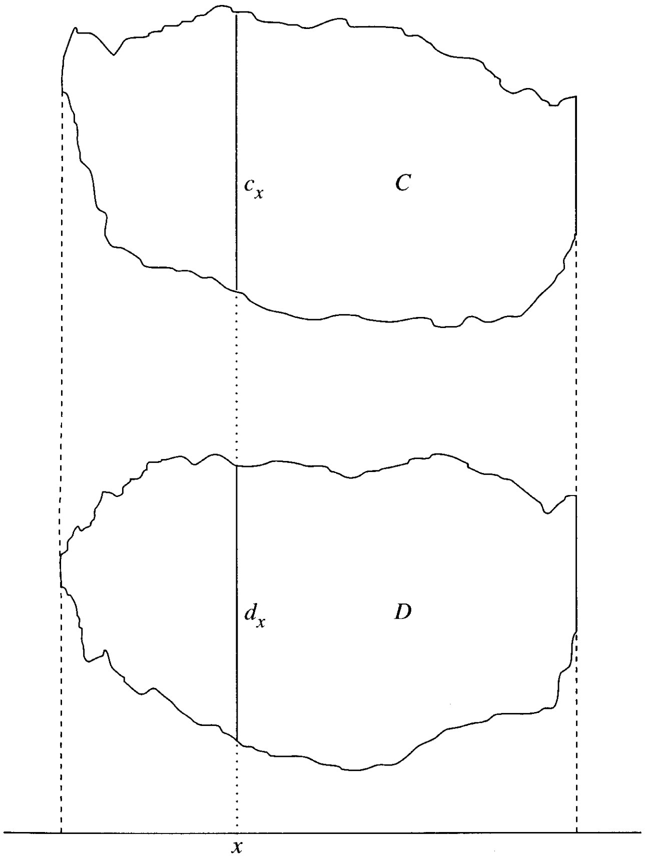


Figure 4.29

Cavalieri's principle: If $d_x = kc_x$ for all x and for a fixed positive number k, then $D = kC$.

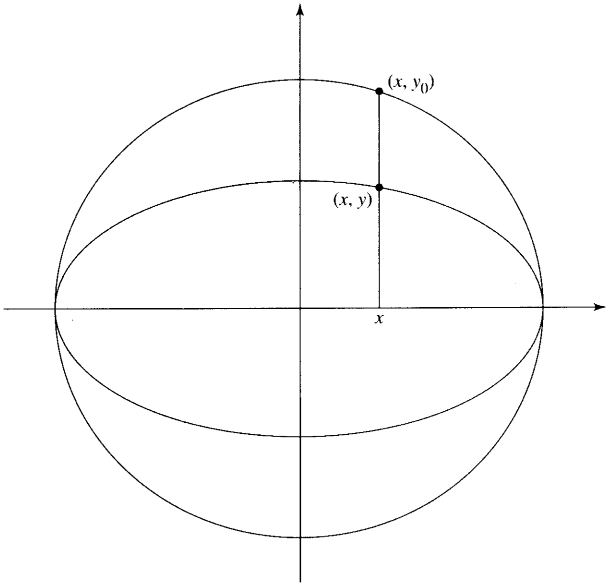
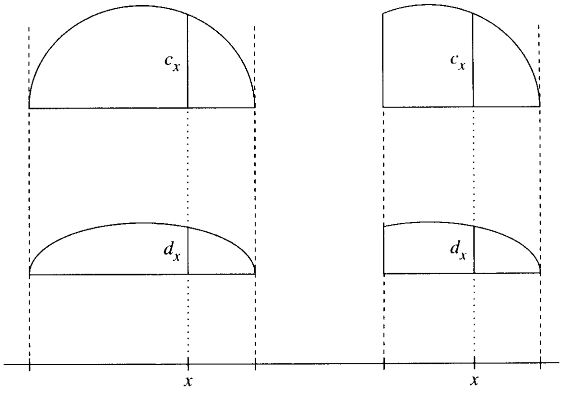


Figure 4.30



(a) (b)

Figure 4.31

Observe, in particular, that if the area C and the constant k can both be determined, then the area D can be determined. As an application of Cavalieri's principle, we will compute the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a > 0$, $b > 0$, and $a \geq b$. Consider simultaneously the graph of the ellipse and that of the circle $x^2 + y^2 = a^2$. Let x satisfy $-a \leq x \leq a$ and, as shown in Figure 4.30, let (x, y) and (x, y_0) be the indicated points on the ellipse and circle, respectively. Since (x, y_0) satisfies $x^2 + y^2 = a^2$ and $y_0 \geq 0$, it follows that $y_0 = \sqrt{a^2 - x^2}$. Because (x, y) is on the ellipse,

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \frac{b^2a^2 - b^2x^2}{a^2} = \frac{b^2}{a^2}(a^2 - x^2).$$

and after taking square roots,

$$y = \frac{b}{a}\sqrt{a^2 - x^2} = \frac{b}{a}y_0.$$

Separate the upper semicircles and the upper part of the ellipse, as shown in Figure 4.31(a). Notice that we just demonstrated that $d_x = \frac{b}{a}c_x$ for any x. Since the area of a semicircle of radius a is $\frac{1}{2}\pi a^2$, it follows by Cavalieri's principle that the area of the upper half of the ellipse is equal to $\frac{b}{a}\left(\frac{1}{2}\pi a^2\right) = \frac{1}{2}\pi ab$. Therefore, the full ellipse with semimajor axis a and semiminor axis b has area

$$\pi ab.$$

Note that Cavalieri's principle also applies to Figure 4.31(b). In particular, the area of the elliptical section has area $\frac{b}{a}$ times that of the semicircular section.

Example 4.14. Refer to Figure 4.32. Consider the section QPN of the ellipse $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$ and the sector $P'ON$ of the circle $x^2 + y^2 = 4^2$. Determine the area of the elliptical section in terms of the angle β.

Solution. The area of the circular sector $P'ON$ is $\frac{1}{2}\beta \cdot 4^2 = 8\beta$. (See Section 3.2.) Since $OP' = 4$, we have $\sin(\pi - \beta) = \frac{QP'}{4}$ and $\cos(\pi - \beta) = \frac{QO}{4}$. Therefore, the area of the triangle $QP'O$ is $\frac{1}{2}QO \cdot QP' = 8\cos(\pi - \beta)\sin(\pi - \beta)$. Adding the area of the triangle to the area of the sector shows that the area of the circular section $QP'N$ is $8\beta + 8\cos(\pi - \beta)\sin(\pi - \beta)$. Since $a = 4$ and $b = 3$, we find by Cavalieri's principle that the area of the elliptical section QPN is

$$\frac{3}{4} \cdot 8\left[\beta + \cos(\pi - \beta)\sin(\pi - \beta)\right]$$
$$= 6\left[\beta + \cos(\pi - \beta)\sin(\pi - \beta)\right].$$

Making use of Example 4.11, we can conclude that $\cos(\pi - \beta) = \cos(\beta - \pi) = -\cos\beta$ and $\sin(\pi - \beta) = -\sin(\beta - \pi) = \sin\beta$. Therefore the area of the elliptical section can be written as $6(\beta - \sin\beta \cdot \cos\beta)$. What happens if the angle β is less than $\frac{\pi}{2}$? Then the area of the circular section $QP'N$ is equal to the area of the circular sector $P'ON$ minus the area of the triangle QOP'. Check that the area of this triangle is $8\sin\beta \cdot \cos\beta$. So the area of the circular section is $QP'N$ is $8\beta - 8\sin\beta \cdot \cos\beta$ and by another application of Cavalieri's principle, the area of the elliptical section QPN is $6\beta - 6\sin\beta \cdot \cos\beta$ in this case also.

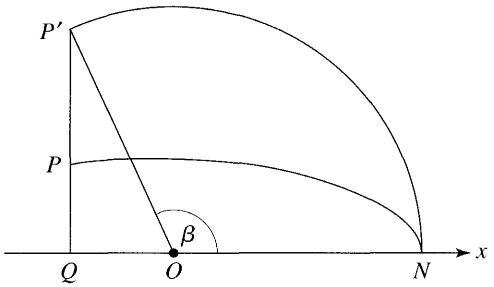


Figure 4.32

4.7 Kepler's Analysis of the Orbits

Let's now turn to Kepler's mathematical analysis of the orbits of the planets. The starting point is Kepler's first law, which states that every planet is in elliptical orbit around the Sun with the Sun at a focal point. The discussion that follows retains the essence of Kepler's work; but it is different in notation as well as in detail, and it adds more recent information.

Recall that one astronomical unit (1 AU) was defined to be equal to the distance between the Earth and the Sun. However, since the Earth's orbit about the Sun is an ellipse, this distance varies, and so more precision is necessary. Let a be the semimajor axis of the Earth's orbit and let e be its linear eccentricity. Since the Sun is at a focus of the ellipse, we see from Figure 4.28 that $a + e$ is the farthest distance from the Earth to the Sun and $a - e$ the closest. So the average distance

$$\frac{(a+e)+(a-e)}{2} = \frac{2a}{2} = a$$

is also equal to the semimajor axis. We will now define 1 AU to be equal to the semimajor axis of

Table 4.2

Orbital Data of Planets

Planet	Semimajor axis in AUs	Period of the orbit in years	Astronomical eccentricity	Angle of orbital plane with that of the Earth	Average speed in miles/sec
Mercury	0.3871	0.2408	0.2056	7.00°	29.6
Venus	0.7233	0.6152	0.0068	3.39°	21.7
Earth	1.0000	1.0000	0.0167	0.00°	18.5
Mars	1.5237	1.8809	0.0934	1.85°	15.0
Jupiter	5.2028	11.8622	0.0484	1.31°	8.1
Saturn	9.5388	29.4577	0.0557	2.49°	6.0

the Earth's orbit. The actual value of 1 AU was not known with any precision until the latter part of the 18th century, when it was shown to be equal to 93,300,000 miles. This was calculated by use of parallax and information about the path that Venus traces out across the Sun. A recent determination by the Jet Propulsion Laboratory in Pasadena, California, is

$$1\ \text{AU} = (499.004784\ \text{seconds}) \times (\text{speed of light})$$
$$= 92{,}955{,}807.2\ \text{miles}.$$

It relies on measuring the round-trip travel time of radio waves (radar) from the Earth to the Sun.

Let's return to Kepler's great adversary, the planet Mars. Tycho Brahe's observations, as well as his own, supplied Kepler with essential quantitative information about the orbit. In his *Astronomia Nova* he recorded the farthest distance from Mars to the Sun as 1.6678 AU and the closest as 1.3850 AU. Let a be the semimajor axis of Mars's ellipse and let e be its linear eccentricity. Refer to Figure 4.28 again and observe that $a + e = 1.6678$ AU and $a - e = 1.3850$ AU. So $2a = 3.0528$ AU and $2e = 0.2828$ AU. Hence $a = 1.5264$ AU and $e = 0.1414$ AU. The astronomical eccentricity of Mars's orbit is $\varepsilon = \frac{e}{a} = 0.0926$. Kepler had precise numerical information about the orbits of the other planets as well.

We will use today's version of these data. See Table 4.2. Only the six planets known at the time of Kepler are listed.

Notice that the astronomical eccentricities are small for all the orbits listed. Since an ellipse with a small astronomical eccentricity is close to being a circle (see Section 4.5), it follows that the orbits of the planets are nearly circular. The orbit of Mercury has the largest astronomical eccentricity and is the most "elliptical."

Example 4.15. Let's continue to look at Mars. Its semimajor axis a is approximately

$$a = (1.5237)(92{,}960{,}000)\ \text{miles}$$
$$= 141.64\ \text{million miles}.$$

Since the linear eccentricity is $e = \varepsilon a$, the semiminor axis b equals

$$\begin{aligned} b &= \sqrt{a^2 - e^2} = \sqrt{a^2 - \varepsilon^2 a^2} \\ &= a\sqrt{1-(0.0934)^2} = a\sqrt{1-0.0087} \\ &= a\sqrt{0.9913} = 0.9956a. \end{aligned}$$

So $b = (0.9956)(141.64) = 141.02$ million miles. Observe that the maximum and minimum distances from Mars to the Sun are, respectively,

$$\begin{aligned} a + e &= a(1+\varepsilon) = a(1 + 0.0934) = 1.0934a \\ &= (1.0934)(141.64) = 154.87 \text{ million miles}, \\ a - e &= a(1-\varepsilon) = a(1 - 0.0934) = 0.9066a \\ &= (0.9066)(141.64) = 128.41 \text{ million miles}. \end{aligned}$$

Example 4.16. The semimajor axis of the Earth is $a = 1$ AU, and its astronomical eccentricity is $\varepsilon = 0.0167$. It follows that the semiminor axis is

$$\begin{aligned} b &= \sqrt{a^2 - \varepsilon^2 a^2} = \sqrt{1-(0.0167)^2} \\ &= \sqrt{1-0.0003} = \sqrt{0.9997} = 0.9998 \text{ AU}. \end{aligned}$$

The distance from the Earth to the Sun varies from a minimum of

$$a - e = a - \varepsilon a = 1 - 0.0167 = 0.9833 \text{ AU}$$

to a maximum of $a + e = 1.0167$ AU. In miles, the variation is from about 91.41 million miles to about 94.51 million miles.

Examples 4.15 and 4.16 show how close the orbits of both the Earth and Mars are to being circles. Note that the semimajor and semiminor axes are nearly equal in each case. It seems remarkable that Kepler was able to conclude that they were not circular. The variations in the maximum and minimum distances from the Sun are significant, however, especially in the case of Mars. Therefore, the issue is not so much the fact that the orbits are elliptical, but rather that the Sun is off-center at a focus. This is the reason, in retrospect, why Ptolemy's off-centered circle simulates the Earth's orbit around the Sun reasonably well (see Section 2.5).

Kepler's first law, in combination with the data of Table 4.2 and the analysis of the ellipse in Section 4.5, gives a precise picture of the shape of the orbits of the planets. Now, however, comes a fundamental question: How exactly, does a given planet trace out its orbit? In other words, where in its orbit is a planet at a given time?

This question can be reformulated with more precision as follows. Refer to Figure 4.33, which depicts a planet in orbit around the Sun, S. Suppose that at a certain time, which we will designate as $t = 0$, the planet is at the perihelion position N, the position closest to the Sun. Let $P = (x, y)$ be the position of the planet at some later time t (in Earth days, for example). Let r be the distance from S to P, and let α be the indicated angle. The problem now is this: can both α and r, and therefore the position P of the planet, be determined in terms of the elapsed time t? Reflecting over Kepler's second law suggests that this should be possible. And indeed it is, as we shall now see! The key to the answer to this problem is a construction that was already employed in the computation of the area of the ellipse. Surrounding the elliptical orbit with a circle (which is purely a mathematical tool with no physical significance) will allow us to separate a difficult problem into manageable components.

Let's begin by placing a Cartesian coordinate system in such a way that the ellipse is in standard position with the focus S at $(e, 0)$ on the positive x-axis. So the equation of the planet's orbit is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where, as before, a and b are the semimajor and semiminor axes, respectively. Recall that $b = \sqrt{a^2 - e^2}$. So $a \geq b, b^2 = a^2 - e^2$ and $e^2 = a^2 - b^2$.

We will now supplement Figure 4.33 as follows. Refer to Figure 4.34. Let the coordinates of the point P be (x, y) and let $X = (x, 0)$ be the point on the x-

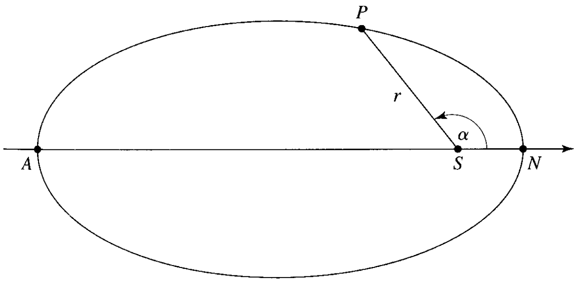


Figure 4.33

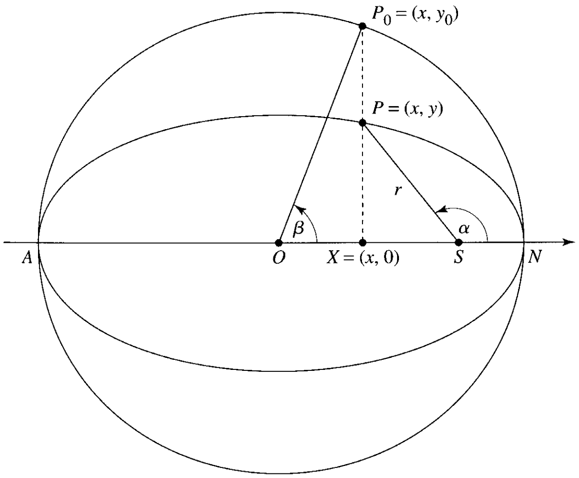


Figure 4.34

axis directly below P. Now superimpose the circle $x^2 + y^2 = a^2$ of radius a, let $P_0 = (x, y_0)$ be the point on the circle directly above P, and let β be the angle determined by ON and OP_0. Finally, let A be the aphelion position of the planet's orbit. Observe that this is the position where its distance from the Sun is greatest.

The solution of the problem requires three steps. Step 1 expresses both r and α in terms of the angle β. Step 2 produces an equation that links β to the time t. Step 3, finally, solves this equation for β in terms of t. The three steps together determine r and α in terms of t. The planet's position P is then given in terms of t, and Kepler's objective is achieved.

Let's turn to Step 1. Note from Figure 4.34 that $\cos\beta = \frac{OX}{OP_0} = \frac{x}{a}$, so that

$$x = a\cos\beta.$$

Similarly, $\sin\beta = \frac{y_0}{a}$, and hence $y_0 = a\sin\beta$. Since $y = \frac{b}{a}y_0$, as already observed in Section 4.6,

$$y = b\sin\beta.$$

Applying the Pythagorean theorem to the triangle SPX and using the equalities $SX = e - x$, $e^2 = a^2 - b^2$ and $\sin^2\beta + \cos^2\beta = 1$, Kepler calculated as follows:

$$\begin{aligned} r^2 &= y^2 + (e - x)^2 \\ &= b^2\sin^2\beta + (e - a\cos\beta)^2 \\ &= b^2\sin^2\beta + a^2\cos^2\beta - 2ae\cos\beta + e^2 \\ &= (a^2 - e^2)\sin^2\beta + a^2\cos^2\beta - 2ae\cos\beta + e^2 \\ &= a^2 - e^2\sin^2\beta - 2ae\cos\beta + e^2 \\ &= a^2 + e^2(1 - \sin^2\beta) - 2ae\cos\beta \\ &= a^2 + e^2\cos^2\beta - 2ae\cos\beta \\ &= (a - e\cos\beta)^2. \end{aligned}$$

Since $a > e \geq e\cos\beta$, the term $a - e\cos\beta$ is positive. Because r is also positive, we can take the square root of both sides of the equality $r^2 = (a - e\cos\beta)^2$ just derived to get $r = a - e\cos\beta$. Factor out a to get $r = a(1 - \frac{e}{a}\cos\beta)$. Recalling that the astronomical eccentricity is $\varepsilon = \frac{e}{a}$, we see that Kepler has derived the equation

$$\boxed{r = a(1 - \varepsilon\cos\beta)}$$

Therefore, r is now determined in terms of β.

By Example 4.12 in Section 4.4,

$$\tan\alpha = -\tan(-\alpha) = -\tan(\pi - \alpha).$$

Another look at the triangle SPX now shows that

$$\tan\alpha = -\tan(\pi - \alpha) = -\frac{y}{e - x} = \frac{b\sin\beta}{a\cos\beta - e}.$$

With the substitution $a\varepsilon = e$, this becomes

$$\tan\alpha = \frac{b\sin\beta}{a(\cos\beta - \varepsilon)}.$$

Thus Kepler has determined $\tan\alpha$, and hence α, in terms of β. It is important to observe that the formulas for both r and $\tan\alpha$ are valid for any position of P along its orbit. This follows from the basic trig formulas of Examples 4.11 and 4.12 of Section 4.4 for each of the three cases (P in the second, third, and fourth quadrants) that must be checked. Refer to Example 4.14, where this was done in a related situation.

Starting with the expression for $\tan\alpha$ that was just determined (and using lots of trigonometry), it is possible to derive the equation

$$\boxed{\tan\frac{\alpha}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\beta}{2}}$$

This formula is easier to use because it involves fewer computations. It is known as Gauss's formula. Its discoverer, the German mathematician-astronomer Carl Friedrich Gauss (1777–1855) is considered to be, along with Archimedes and Newton, the most famous and brilliant of all mathematicians.[4] Exercises 4F will "walk you through" the derivation of this formula.

Step 2 follows next. Recall that its purpose is to connect the angle β with the elapsed time t. We will see that this connection is a consequence of Kepler's second law.

Let A_t be the area swept out by the segment SP during time t the planet moves from N to P. See Figure 4.35. Step 2 begins with the computation of A_t in terms of β.

Refer back to Figure 4.34 and note that

$$\begin{aligned}&\text{Area section } P_0XN\\ &\quad = \text{Area sector } P_0ON - \text{Area } \Delta P_0XO\end{aligned}$$

Because P_0ON is a circular sector with radius a and angle β, its area is $\frac{1}{2}a^2\beta$. See Section 3.2. Since the area of the right triangle P_0XO is $\frac{1}{2}xy_0$ and $y_0 = a\sin\beta$, it follows that

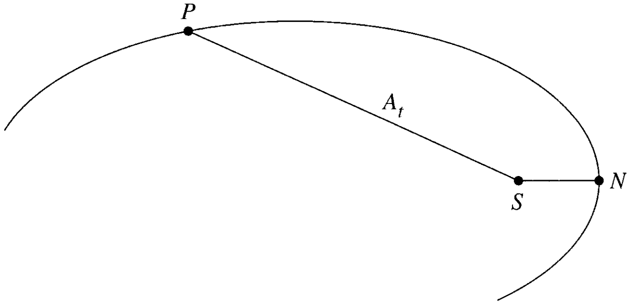


Figure 4.35

$$\text{Area section } P_0XN = \frac{1}{2}\beta a^2 - \frac{1}{2}xa\sin\beta.$$

Refer next to Section 4.6, in particular to Figure 4.31(b) and deduce that the elliptical section PXN has area $\frac{b}{a}$ times that of the section P_0XN just computed. Therefore,

$$\begin{aligned}\text{Area section } PXN &= \frac{b}{a}\left(\frac{1}{2}\beta a^2 - \frac{1}{2}xa\sin\beta\right)\\ &= \frac{1}{2}\beta ab - \frac{1}{2}xb\sin\beta.\end{aligned}$$

Now return to A_t and observe that

$$\begin{aligned}A_t &= \text{Area section } PXN - \text{Area } \Delta PXS\\ &= \left(\frac{1}{2}\beta ab - \frac{1}{2}xb\sin\beta\right) - \frac{1}{2}(e-x)y\\ &= \left(\frac{1}{2}\beta ab - \frac{1}{2}xb\sin\beta\right) - \frac{1}{2}(e-x)b\sin\beta\\ &= \frac{1}{2}\beta ba - \frac{1}{2}eb\sin\beta = \frac{1}{2}\beta ab - \frac{1}{2}\varepsilon ab\sin\beta\\ &= \frac{ab}{2}(\beta - \varepsilon\sin\beta).\end{aligned}$$

The quantity A_t can be computed in another way. As consequence of Kepler's second law (refer to Section 4.1), the ratio $\frac{A_t}{t}$ is the same for any t. It can therefore be evaluated by taking t to be the period T of the orbit of the planet. Since the area of the full ellipse is $ab\pi$, it follows that $\frac{A_t}{t} = \frac{ab\pi}{T}$. So $A_t = \frac{ab\pi t}{T}$. By setting the two expressions for A_t equal to each other, $\frac{ab}{2}(\beta - \varepsilon \sin \beta) = \frac{ab\pi t}{T}$. After a simplification,

$$\boxed{\beta - \varepsilon \sin \beta = \frac{2\pi t}{T}}$$

This is a famous equation, known appropriately as Kepler's equation. It provides the link between t and β. It should be noted that Kepler's equation and its derivation are valid for any position P of the planet and any elapsed time t (even though our argument relies on Figures 4.34 and 4.35, which place the planet in the first quadrant of its first orbit). Step 2 is now complete.

The problem that remains is the solution of Kepler's equation for β in terms of t. This objective of Step 3 is taken up in the next section.

4.8 The Method of Successive Approximations

To solve the equation $\beta - \varepsilon \sin \beta = \frac{2\pi t}{T}$ for β is not easy. To do so, we will use the method of *successive approximations*. If you have ever determined your weight on a scale that has a sliding pointer, then you already understand the essence of this method. The start is made with an educated guess (an initial estimate of your weight), and then an approximation step (slide the pointer) is used to successively close in on the desired value (your weight). The game "played with squares" in Section 3.3 is also an example of the method of successive approximations. There the educated guess (not so educated, actually) is the choice of the black L as a first approximation of the area of the square. The approximation step is this: reduce the dimension of L by a factor of four and insert this reduced L into the white square that remains. Applied over and over, this procedure homes in on the area of the square.

We will illustrate the method of successive approximations in another context. Suppose that $p(x)$ is a polynomial. For example, take

$$p(x) = x^4 + 2x^3 - 12x - 1.$$

Notice that $p(0) = -1$ and that

$$p(2) = 2^4 + 2 \cdot 2^3 - 12 \cdot 2 - 1 = 16 + 16 - 24 - 1 = 7.$$

Since $p(0)$ is negative and $p(2)$ is positive, it follows that $p(c) = 0$ for some number c between 0 and 2. Is there a way to determine c? Figure 4.36 illustrates how this can be done.

1. Make an educated guess, say c_1, for c.
2. Use the following approximation step: from a point on the x-axis go up to the graph of $p(x)$ and down on the tangent to a new point on the x-axis.

Begin by applying the approximation step to the guess c_1 to get c_2; apply the approximation step to c_2 to get c_3; apply it to c_3 to get $c_4, \ldots$. Continue in this way, and observe that the sequence of numbers $c_1, c_2, c_3, c_4, \ldots$ zeroes in on the root c. This method of determining the root of a polynomial is known as *Newton's method*. It can be made numerically precise (Exercise 8K of Chapter 8 describes how this is done), but this is not the point now.

We will now turn to Step 3 in the determination of the position of the planet in its orbit after a given

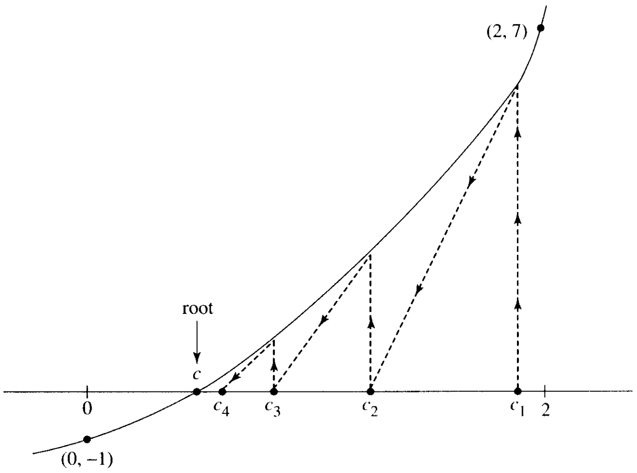


Figure 4.36

elapsed time t. The task in this final step is to consider the time t as given and to solve the equation

$$\beta - \varepsilon \sin \beta = \frac{2\pi t}{T}$$

for β in terms of t. A key ingredient in the solution is the inequality

$$|\sin \varphi - \sin \theta| \leq |\varphi - \theta|$$

for any two angles φ and θ in radian measure. The validity of this inequality ultimately depends on calculus (it will be verified in Section 8.7), but it can be made plausible as follows. Let $(\theta, \sin \theta)$ and $(\varphi, \sin \varphi)$ be any two points on the graph of the sine. Figure 4.37 shows two such pairs of points. In each case, let v and h be the vertical and the horizontal distance between the points. Since the graph of the sine is rather flat, it turns out that the vertical distance v is less than or equal to the horizontal distance h. This is so regardless of where the two points are taken on the graph. Therefore $v \leq h$ in all cases. From Section 4.3 we know that $v = |\sin \varphi - \sin \theta|$ and $h = |\varphi - \theta|$. Therefore,

$$|\sin \varphi - \sin \theta| \leq |\varphi - \theta|$$

for any θ and φ.

We now solve $\beta - \varepsilon \sin \beta = \frac{2\pi t}{T}$ for β by the method of successive approximations.

1. The first stab at β is $\beta_1 = \frac{2\pi t}{T}$. Since $|\beta - \beta_1| = \left|\beta - \frac{2\pi t}{T}\right| = |\varepsilon \sin \beta| \leq \varepsilon$ and because the astronomical eccentricity $\varepsilon = \frac{e}{a}$ is small (see Table 4.2), β_1 approximates β. So $\beta_1 = \frac{2\pi t}{T}$ is a good educated guess.
2. The approximation step is this: place an angle (in radians) into the "blank" of the expression $\frac{2\pi t}{T} + \varepsilon \sin(\quad)$ to get a new angle (in radians).

Start by putting $\beta_1 = \frac{2\pi t}{T}$ into the blank. This gives the new angle $\beta_2 = \frac{2\pi t}{T} + \varepsilon \sin \beta_1$. Repeating (2) with β_2, we get $\beta_3 = \frac{2\pi t}{T} + \varepsilon \sin \beta_2$. Doing this again and again, we get $\beta_4 = \frac{2\pi t}{T} + \varepsilon \sin \beta_3$, $\beta_5 = \frac{2\pi t}{T} + \varepsilon \sin \beta_4, \ldots,$ and $\beta_i = \frac{2\pi t}{T} + \varepsilon \sin \beta_{i-1}, \ldots$. The big question is this: Does the sequence of numbers

$$\beta_1,\ \beta_2,\ \beta_3, \ldots,\ \beta_i, \ldots$$

close in on the solution β of $\beta - \varepsilon \sin \beta = \frac{2\pi t}{T}$? This is indeed the case. Notice that $\beta = \frac{2\pi t}{T} + \varepsilon \sin \beta$. It follows that

$$\begin{aligned}\beta - \beta_2 &= \left(\frac{2\pi t}{T} + \varepsilon \sin \beta\right) - \left(\frac{2\pi t}{T} + \varepsilon \sin \beta_1\right)\\ &= \varepsilon(\sin \beta - \sin \beta_1).\end{aligned}$$

Therefore, using (1),

$$|\beta - \beta_2| = \varepsilon |\sin \beta - \sin \beta_1| \leq \varepsilon |\beta - \beta_1| \leq \varepsilon^2.$$

In the same way,

$$\begin{aligned}\beta - \beta_3 &= \left(\frac{2\pi t}{T} + \varepsilon \sin \beta\right) - \left(\frac{2\pi t}{T} + \varepsilon \sin \beta_2\right)\\ &= \varepsilon(\sin \beta - \sin \beta_2),\end{aligned}$$

so that

$$|\beta - \beta_3| = \varepsilon |\sin \beta - \sin \beta_2| \leq \varepsilon |\beta - \beta_2| \leq \varepsilon^3.$$

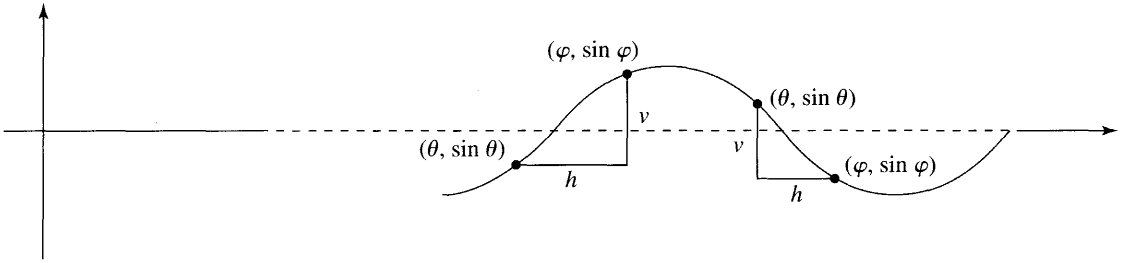


Figure 4.37

Repeating this computation again and again shows that

$$|\beta-\beta_4|\le\varepsilon^4,\ |\beta-\beta_5|\le\varepsilon^5,\ldots,\ |\beta-\beta_i|\le\varepsilon^i,\ldots.$$

Since ε is small, the powers ε^2, ε^3, $\varepsilon^4,\ldots$ become progressively smaller. Therefore, the distances $|\beta-\beta_1|$, $|\beta-\beta_2|$, $|\beta-\beta_3|,\ldots$ between $\beta_1, \beta_2, \beta_3,\ldots$ and β get progressively smaller. So the numbers $\beta_1, \beta_2, \beta_3,\ldots$ close in on β as required. Since $0\le\varepsilon<1$ for any ellipse, this successive approximation process will always converge to a solution β. If ε is small (and this is so for all the planets) it will converge very quickly.

Let's consider the case of the Earth. Here

$$\varepsilon=0.0167,\ \varepsilon^2=0.000279,$$

$$\varepsilon^3=0.00000467,\ldots,\ \varepsilon^6=2.17\times10^{-11}.$$

Since $|\beta-\beta_6|\le\varepsilon^6<10^{-10}$, the sixth approximation β_6 of β is already an extremely good estimate of β.

4.9 Computing Orbital Information

We will now compute the position of the Earth at vernal equinox and at summer solstice of 1994 using the method developed in Sections 4.7 and 4.8. All of the data we use was supplied by the U.S. Naval Observatory. We will work with an accuracy of four decimal places. The Earth's orbit is $T=365.2422$ days and the four seasons have the following lengths:

Spring: 92 days, 18 hours, 20 minutes = 92.7639 days.
Summer: 93, 15 hours, 31 minutes = 93.6465 days.
Autumn: 89 days, 20 hours, 4 minutes = 89.8361 days.
Winter: 88 days, 22 hours, 54 minutes = 88.9938 days.

In 1994, perihelion occurred on January 2 at 5 hours 55 minutes Greenwich Mean Time (GMT).[5] This is $t=0$. Since vernal equinox occurred that year on March 20 at 20 hours 28 minutes, the elapsed time in days from perihelion to vernal equinox for 1994 was

$$t_{ve}=77\text{ days, }14\text{ hours, }33\text{ minutes}=77.6023\text{ days.}$$

Adding the 92.7639 days of spring to this shows that the elapsed time to summer solstice is

$$t_{ss}=77.6023\text{ days}+92.7639\text{ days}=170.3662\text{ days.}$$

Adding the lengths of summer and autumn gives the elapsed times to autumnal equinox and winter solstice as $t_{ae}=265.0132$ days and $t_{ws}=354.8493$ days, respectively.

To compute the position of the Earth at vernal equinox, we will determine r and α for $t=t_{ve}=77.6023$. The strategy has already been laid out. First, determine β for $t=77.6023$, and then substitute this value into the equations

$$r=a(1-\varepsilon\cos\beta)\quad\text{and}\quad\tan\frac{\alpha}{2}=\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\beta}{2}.$$

So the first step is to solve $\beta-\varepsilon\sin\beta=\frac{2\pi t}{T}$ for β. Since $t=t_{ve}=77.6023$,

$$\frac{2\pi t}{T}=\frac{2\pi(77.6023)}{365.2422}=\frac{(6.2832)(77.6023)}{365.2422}=1.3350.$$

The solution of $\beta-\varepsilon\sin\beta=1.3350$ by successive approximation proceeds as follows: $\beta_1=\frac{2\pi t}{T}=1.3350$. Successive substitution into $\frac{2\pi t}{T}+\varepsilon\sin(\ \ )$ starting with $\beta_1=\frac{2\pi t}{T}$ gives

$$\beta_2=\frac{2\pi t}{T}+\varepsilon\sin\beta_1=1.3350+(0.0167)(\sin 1.3350)$$
$$=1.3350+(0.0167)(0.9723)=1.3350+0.0162$$
$$=1.3512,$$

$$\beta_3=\frac{2\pi t}{T}+\varepsilon\sin\beta_2=1.3350+(0.0167)(\sin 1.3512)$$
$$=1.3350+(0.0167)(0.9760)=1.3350+0.0163$$
$$=1.3513,$$

$$\beta_4=\frac{2\pi t}{T}+\varepsilon\sin\beta_3=1.3350+(0.0167)(\sin 1.3513)$$
$$=1.3350+(0.0167)(0.9760)=1.3350+0.0163$$

$$= 1.3513.$$

Since the sequence of betas has stabilized at 1.3513, this is the value of β within the four decimal accuracy with which we are working. The value $\beta = 1.3513$ must be placed into the equations

$$r = a(1 - \varepsilon \cos \beta) \quad \text{and} \quad \tan\frac{\alpha}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\beta}{2}$$

to get r and α for the vernal equinox position. Since $a = 1$ AU,

$$\begin{aligned} r &= 1 - (0.0167)(\cos 1.3513) \\ &= 1 - (0.0167)(0.2177) = 1 - 0.0036 \\ &= 0.9964 \text{ AU} \end{aligned}$$

and

$$\begin{aligned} \tan\frac{\alpha}{2} &= \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\beta}{2} \\ &= (\tan 0.6757)\sqrt{\frac{1.0167}{0.9833}} = (1.0168)(0.8016) \\ &= 0.8151. \end{aligned}$$

Doing an inverse tangent operation with the calculator shows that $\frac{\alpha}{2} = 0.6839$. So $\alpha = 1.3678$ radians, or 78.3690°.

Are our conclusions reasonable? The length of the year is about 365.25 days. So during an elapsed time of $t = 77.6$ days from its perihelion position, the Earth should trace out roughly $\frac{77.6}{365.25} \times 360 = 76.5^\circ$. Therefore, α ought to be close to 76.5°. Since the Earth moves faster near perihelion, it should be somewhat more than this. And it is! Consider the fact that α is less than 90° and refer back to Figure 4.27. Note that $r = 0.9964$ AU ought to be less than the semimajor axis b of the Earth's orbit. And it is! By Example 4.16, $b = 0.9998$ AU.

Now on to summer solstice. In this case $t = t_{ss} = 170.3662$ days. Solving $\beta - \varepsilon \sin \beta = \frac{2\pi t}{T}$ for β this time, start with

$$\beta_1 = \frac{2\pi t}{T} = \frac{(6.2832)(170.3662)}{365.2422} = 2.9308.$$

Computing as before, we get

$$\begin{aligned} \beta_2 &= \frac{2\pi t}{T} + \varepsilon \sin \beta_1 \\ &= 2.9308 + (0.0167)(\sin 2.9308) = 2.9343, \\ \beta_3 &= \frac{2\pi t}{T} + \varepsilon \sin \beta_2 = 2.9342, \quad \text{and} \\ \beta_4 &= \frac{2\pi t}{T} + \varepsilon \sin \beta_3 = 2.9342. \end{aligned}$$

Therefore, $\beta = 2.9342$.

Substituting $\beta = 2.9342$ into $r = 1 - \varepsilon \cos \beta$ gives us

$$\begin{aligned} r &= 1 - (0.0167)(\cos 2.9342) = 1 + (0.0167)(0.9786) \\ &= 1.0163 \text{ AU.} \end{aligned}$$

This is close to the maximum of 1.0167 AU. Since in 1994 summer solstice occurred on June 21 and aphelion on July 5, this is to be expected. This also suggests that α should be close to 180°. Substituting $\beta = 2.9342$ into $\tan\frac{\alpha}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\beta}{2}$, we get

$$\begin{aligned} \tan\frac{\alpha}{2} &= (\tan 1.4671)\sqrt{\frac{1.0167}{0.9833}} = (1.0168)(9.690) \\ &= 9.8528. \end{aligned}$$

Now $\frac{\alpha}{2} = 84.2047^\circ$, so $\alpha = 168.4094^\circ$.

Lean back from this book and think about what Kepler has achieved. Think about Mars in orbit about the Sun. It is over one hundred million miles from Earth. It is visible on some clear nights, but only as a minuscule point of light. And Kepler has determined exactly how it moves!! He did so with an impressive combination of careful observations, the formulations of the fundamental laws, and very delicate mathematics. Incidentally, Kepler's mathematics can be fed into a desktop computer to provide a simulation of the motion of the planets in their elliptical orbits.[6]

4.10 Postscript

Recall that one of the principal goals set out in this chapter was the extraction of precise quantitative information about the orbit of a planet from Kepler's first two laws. This has now been accomplished. In sharp contrast with the systems of both Ptolemy and Copernicus, only a single theory is required. This theory applies not only to every planet, but also to comets and asteroids around the Sun. Indeed, it applies to Jupiter and its satellites. Analytic geometry plays a prominent role, but calculus—in the guise of Cavalieri's principle and the important inequality $|\sin\varphi - \sin\theta| \le |\varphi - \theta|$—is present also. Put more explicitly, the rigorous justifications of both Cavalieri's principle and this inequality require—as we shall see—the methods of calculus.

Another important question that Kepler grappled with, but ultimately did not solve, concerns the driving force that keeps a planet in its orbit. Kepler realized that the Sun plays a role and suspected that such a force diminishes with distance. Indeed, it is a consequence of his third law that the average velocities of the planets fall off as their distances from the Sun increase (see Table 4.2). However, he was unable to determine the precise quantitative measure of this force. This and related matters remain—as we shall see—for Isaac Newton to accomplish.

A more immediate question concerns a matter as yet untouched, namely, the observation of Galileo that the rate of increase

$$\frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

of the velocity of a falling body is a constant. What quantitative information does this fact contain and how can it be brought to the surface? This too requires the methods of calculus.

The time is ripe. The Italians Cavalieri and Torricelli; the Frenchmen Descartes, Fermat, Pascal, and Roberval; the Dutchman Huygens; the Englishmen Willis and Barrow; Gregory from Scotland; and a number of other mathematicians have set the stage and have put many of the components into place. It remains for Newton and Leibniz to add some decisive touches but, more importantly, to synthesize special cases and examples into a unified theory and into an integrated system that can be applied to a wide variety of mathematical problems: The Differential and Integral Calculus.

Exercises
4A. Basic Analytic Geometry

1. Find the distance between the points (1,1) and (4,5), and then between $(1, -6)$ and $(-1, -3)$.

2. Consider the triangle with vertices $A = (6, -7)$, $B = (11, -3)$, and $C = (2, -2)$.

  i. Show that it is a right triangle by using Pythagorean theorem.
  ii. Find the area of the triangle.

3. Show that the points $A = (-1, 3)$, $B = (3, 11)$, and $C = (5, 15)$ all lie on the same line by showing that $AB + BC = AC$.

4. Sketch the graphs of the equations

  i. $x = 3$
  ii. $y = -2$

5. Sketch the graphs of

  i. $xy = 0$
  ii. $|y| = 1$

In Exercise 6–8, sketch the given region in the x-y plane.

6. $\{(x, y) \mid xy < 0\}$

7. $\{(x, y) \mid 0 \le y \le 4 \text{ and } x \le 2\}$

8. $\{(x, y) \mid |x| < 3 \text{ and } |y| < 2\}$

9. Show that the midpoint of the line segment for $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

10. Find the midpoint of the line segment joining the given points:

  i. $(1, 3)$ and $(7, 15)$
  ii. $(-1, 6)$ and $(8, -12)$

4B. Circles, Parabolas, and Ellipses

In Exercises 11–15, identify the graph of the equation. Is it a parabola, circle, or ellipse? Sketch the graph in each case.

[Hint: For some problems it will be helpful to complete the square.]

11. $y = -x^2 + 3x + 4$

12. $x^2 + 4y^2 = 16$

13. $x = 2y^2$

14. $(x-3)^2 + (y+5)^2 = 7$

15. $9x^2 + 2y^2 = 12$

16. Sketch the graphs of the parabolas $y = -x^2$ and $y = x^2 + 1$. Then shade in the region R in the plane given by $R = \{(x, y) \mid -x^2 \leq y \leq x^2 + 1\}$.

17. Consider the parabolic section determined by the parabola $y = 3x^2 + 6x + 7$ and the line $y = 8$. Let $P = (x, y)$ be any point in the plane. What are the conditions on the coordinates x and y to guarantee that P lies in the parabolic section?

18. Consider the parabola $y = x^2 + 4x + 7$ and find the coordinates of the lowest point. Cut the parabola with the line $y = 7$ and compute the area of the parabolic section obtained.

19. Determine the directrix and focus of the parabola $y = 3x^2 - 2x + 5$.

20. Find an equation of a circle that has center $(3, -1)$ and radius 5.

21. Show that the graph of the equation

$$x^2 + y^2 - 4x + 10y + 13 = 0$$

is a circle. Find its center and radius.

22. Under what condition on the coefficients a, b, and c does the equation $x^2 + y^2 + ax + by + c = 0$ represent a circle? When this condition is satisfied, find the center and radius of the circle.

23. Find the semimajor axis, the semiminor axis, and the linear and astronomical eccentricities of the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$.

24. To draw an ellipse with focal points F_1 and F_2, semimajor axis a and semiminor axis b, proceed as follows. Notice that the distance between the focal points is $2\sqrt{a^2 - b^2}$. Tie a string into a loop of length $2a + 2\sqrt{a^2 - b^2}$. Take a board, place thumbtacks at F_1 and F_2, and put the loop around the tacks. Take a pencil and stretch the string taut. Keep the string stretched and draw a complete revolution. Explain why this results in an ellipse and why it has semimajor axis a and semiminor axis b.

25. Fix a Cartesian coordinate system. Take a line segment of fixed length and let P be a fixed point on it. Slide it around in the plane over all positions with the property that one endpoint is always on the x-axis and the other on the y-axis. Show that in the process the point P describes an ellipse.

26. Place a hyperbola in the x-y plane in such a way that the focal points F_1 and F_2 are on the x-axis and the origin is the midpoint of the segment F_1F_2. Show that the equation of this hyperbola has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. [Hint: Refer to Section 4.5 and follow the development of the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for the ellipse.]

4C. Some Geometry and Trigonometry

27. Refer to Figure 4.19 in Section 4.4. Suppose that $r = 1$ and $s = 0.693$. What is the radian measure of θ and what are the coordinates of the point P?

28. Suppose that in Figure 4.19 of Section 4.4, $r = 2$ and $\theta = 5$ radians. What is the length s of the arc?

29. Refer to Figure 4.20 in Section 4.4. Consider the angles $\theta = 17.52$ and $\theta = -21.83$. In each case place the point P_θ carefully on the unit circle. What are the coordinates of the point in each case?

30. The trigonometric function secant is defined by $\sec\theta = \frac{1}{\cos\theta}$. Sketch the graph of this function by analyzing the graph of the cosine. Refer to Figure 4.25 in Section 4.4.

31. Verify the identities $\sec(\theta + 2\pi) = \sec\theta$, $\sec(-\theta) = \sec\theta$, $\sec(\theta + \pi) = -\sec\theta$, and the formula $\sec^2\theta = \tan^2\theta + 1$.

4D. Computing Orbital Information

32. Determine the following quantities for the orbit of Jupiter: the linear eccentricity, semiminor axis, and the greatest and least distances from the Sun. Do so first in AUs and then convert to miles. (Use the data of Table 4.2.)

33. Use the data in Table 4.2 to verify Kepler's third law for Mars, Jupiter, and Saturn. Why are all these ratios equal to 1?

34. Consider a planet (or a comet) in elliptical orbit around the Sun S, as pictured in Figure 4.38. We already know that its speed is greater when it is nearer the Sun than when it is farther away. Can we say more? Observe the planet from P to P' and then again from A to A'. In the figure, arc PP' has the perihelion position as its midpoint, and arc AA' has the aphelion position as its midpoint. Suppose that both arcs are relatively small and regard the regions PSP' and ASA' to be sectors of a circle. Notice that their radii are $a - e$ and $a + e$, respectively where a is the semimajor axis and e the linear eccentricity of the ellipse.

- **i.** Use the formula for the area of a circular sector (from Section 3.2) to show that the areas of the two sectors are $\frac{1}{2}(a - e)(\text{arc}PP')$ and $\frac{1}{2}(a + e)(\text{arc}AA')$, respectively.
- **ii.** Suppose that the time it takes the body to travel from P to P' is the same as that from A to A'. Let v_P be the average velocity of the body from P to P' and let v_A be its average velocity from A to A'. Show that $\frac{v_P}{v_A} = \frac{a+e}{a-e}$. What is this ratio for the Earth? For Saturn?

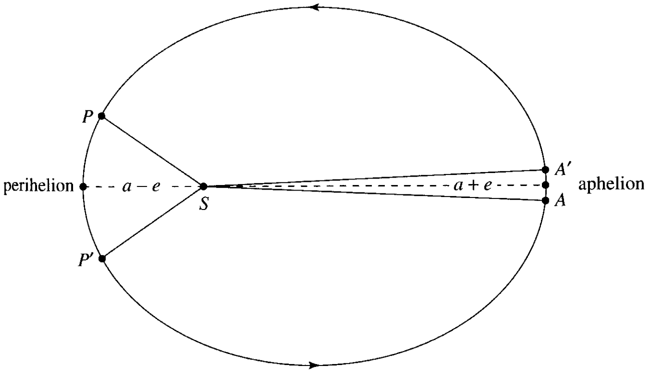


Figure 4.38

35. Do the analysis of Section 4.9 with the data from 1995. In that year perihelion occurred on January 4 at 11 hours 5 minutes GMT, and vernal equinox was on March 21 at 2 hours 14 minutes. Refer to the text for the lengths of the seasons.

- **i.** Determine t_{ve} and t_{ss}.
- **ii.** Determine β for t_{ve}.
- **iii.** Substitute into the formulas $r = a(1 - \varepsilon \cos \beta)$ and $\tan \frac{\alpha}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \tan \frac{\beta}{2}$ to compute r in AUs and α in radians. (Note that $a = 1$, since the Earth is being considered.)
- **iv.** Repeat steps (ii) and (iii) for t_{ss}.

36. Suppose $\varepsilon = 0$ in the analysis of Sections 4.7 and 4.8. Explain why the following are true: the ellipse and the circle in Figure 4.34 coincide, $S = O$, $\alpha = \beta$, $r = a$, and $\beta = \frac{2\pi t}{T}$. Is successive approximation needed?

4E. The Orbit of Halley's Comet

In 1705, Edmund Halley, an astronomer and a contemporary of Isaac Newton, published calculations about the orbits of the comets.[7] He noted that properties of the orbits of the bright comets of 1531, 1607, and 1682 were so similar that the three comets could well be the same. This was later verified, and the comet was named in Halley's honor. The elliptical orbit of Halley's comet has an average period of 76 years. (In fact, the period varies from 74 to 78 years due to irregularities caused by the gravitational attraction of Jupiter and Saturn.) The last time Halley (we will refer to the comet simply as Halley, as confusion with the astronomer is unlikely) was near the Earth was in 1986. The minimal distance of Halley from the Sun, i.e., the distance at perihelion, is known to be $d = 0.59$ AU. The unit of distance throughout Exercises 37–39 is the AU.

Let a and b be the semimajor and semiminor axes of Halley's elliptical orbit and draw the ellipse on an x-y plane such that the equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. See Figure 4.39. The Sun S is at a focal point.

37. Use Kepler's third law with Halley and the Earth to compute a in AUs. Use the fact that $d = 0.59$ AU to compute the linear eccentricity e of Halley's orbit and the semiminor axis b. What is Halley's astronomical eccentricity ε? What is the greatest distance from Halley to the Sun? Refer to Exercise 34ii and compute the ratio of the velocities for Halley.

Figure 4.40 is an enlargement of Halley's orbit near the Sun S. (Why is $S = (e, 0)$?) It also shows the orbit of the Earth. For Exercises 38 and 39 the Earth's orbit is taken to be a circle of radius 1 AU and center the Sun S. In the figure it is assumed that the orbits of Halley and the Earth lie in the same plane. The points H_1 and H_2 are the points of intersection of the two orbits, and x is their common x-coordinate.

38. Determine an equation of the Earth's circular orbit. Show that the x-coordinate of the points of intersection H_1 and H_2 is $x = \frac{a^2-a}{e}$. Determine the numerical values of the x- and y-coordinates of the points H_1 and H_2. Does Figure 4.40 give an accurate picture of Halley's path inside the Earth's orbit? [Hints: In reference to the equation of the ellipse, it is better to work with the parameters a, b, e, etc., rather than their numerical values. Use the identity $e^2 = a^2 - b^2$. Since $\frac{a^2+a}{e} = \frac{a^2}{e} + \frac{a}{e} > a$, note that $x = \frac{a^2+a}{e}$ is not possible.]

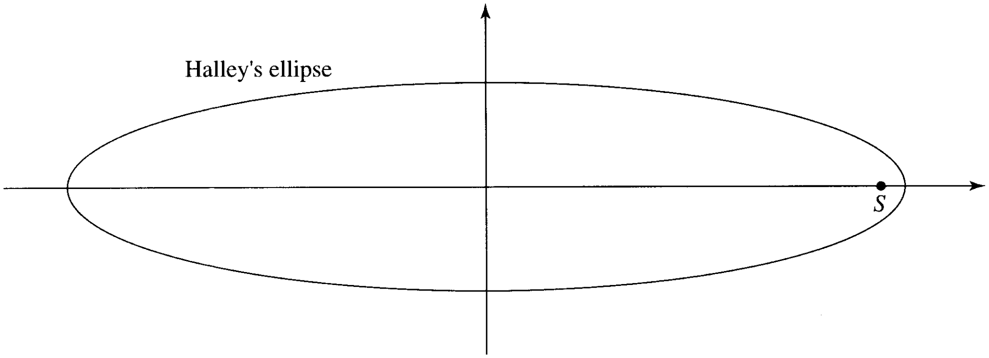


Figure 4.39

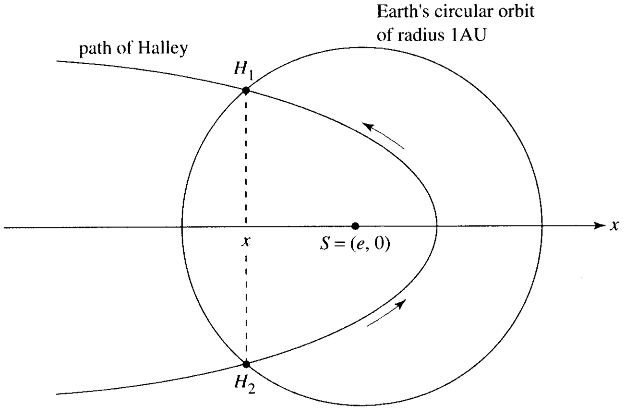


Figure 4.40

39. Consider Figure 4.34 in Section 4.7 in the context of Halley's orbit and take $P = H_1$. What is r equal to? Use the formulas of Sections 4.7 and 4.8 to compute β and t (in years). For how many days will Halley be inside the Earth's orbit?

40. Consider the successive approximation method in reference to computing an angle β for Halley's orbit. Give an estimate for the number of iterations necessary to do so with an accuracy of 0.0002. [Hint: Keep squaring ε with a calculator.]

41. Use the method of Exercise 24 to make a sketch of the elliptical orbit of Halley's comet on a "legal" size sheet of paper. (Use the scale 1 AU = $\frac{3}{8}$ inch.) Use the data from Table 4.2 of Section 4.7 and give indication in your sketch of the location of the orbits of the planets (draw them as circles or parts thereof).

4F. More Trigonometry and Gauss's Formula

Verify Gauss's equation $\tan\frac{\alpha}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\beta}{2}$ with the following sequence of steps:

42. Start with the equation $\tan\alpha = \frac{b\sin\beta}{a\cos\beta - e}$ that was already established. Square both sides; add 1 to both sides; take common denominators; and use the identities $a^2 = b^2+e^2$, $\sin^2\beta+\cos^2\beta = 1$, $\tan^2\beta+1 = \sec^2\beta$, and $\sec\alpha = \frac{1}{\cos\alpha}$ to get

$$\cos^2\alpha = \frac{(a\cos\beta - e)^2}{(a - e\cos\beta)^2}.$$

Because $a - e\cos\beta$ is always positive (why?) and $a\cos\beta - e = x - e$ and $\cos\alpha$ have the same sign (study Figure 4.34 in Section 4.7 and the graph of the cosine carefully), conclude that

$$\cos\alpha = \frac{a\cos\beta - e}{a - e\cos\beta}.$$

Since $\varepsilon = \frac{e}{a}$, you now get

$$\cos\alpha = \frac{\cos\beta - \varepsilon}{1 - \varepsilon\cos\beta}.$$

43. Take $\theta = \varphi$ in the formula

$$\cos(\varphi + \theta) = (\cos\varphi)(\cos\theta) - (\sin\varphi)(\sin\theta)$$

to get $\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$. Deduce the formulas $2\cos^2\varphi = 1 + \cos 2\varphi$ and $2\sin^2\varphi = 1 - \cos 2\varphi$.

44. Deduce that

$$\tan^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{1+\varepsilon}{1-\varepsilon}\frac{1-\cos\beta}{1+\cos\beta}$$
$$= \frac{1+\varepsilon}{1-\varepsilon}\tan^2\frac{\beta}{2}$$

and therefore that $\tan\frac{\alpha}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\beta}{2}$.

4G. A Study of Kepler's Formulas

45. Redraw Figure 4.34 of Section 4.7, this time with P in the third quadrant instead of the first. Study the derivations of the formulas $r = a(1 - \varepsilon\cos\beta)$, $\tan\alpha = \frac{b\sin\beta}{a(\cos\beta - e)}$ and $\beta - \varepsilon\sin\beta = \frac{2\pi t}{T}$ of Section 4.7 and adapt them to this situation. [Hint: Use the trig formulas of Examples 11 and 12, and refer to Example 14.]

46. Consider Figure 4.34 with P as shown in the first quadrant, but suppose that the planet is in its second orbit at time t. Explain why the formulas referred to in Exercise 45 are valid in this case also.

Notes

[1] "Without the concepts, methods and results found and developed by previous generations right down to Greek antiquity, one cannot understand either the aims or the achievements of mathematics in the last fifty years." Thus wrote Hermann Weyl (1885–1955), one of the leading mathematicians of the 20th century.

[2] The word oval comes from the Latin word *ovum*, meaning egg. The "lengthwise" cross-section of most eggs is a curve that is "rounder" at one end and "more pointed" at the other. Such a curve is what Kepler had in mind. Aphelion and perihelion are the positions at which a planet or comet (in this case Mars) are, respectively farthest and closest to the Sun.

[3] For the details about this and other experiments of Galileo, see Stillman Drake, *Galileo: Pioneer Scientist*, University of Toronto Press, Toronto, 1990. It appears that Galileo did not measure g explicitly. After Galileo's time, g was generally thought to be about 24 ft/sec^2. Later, in the 17th century, the Dutch scientist Huygens measured g to be the equivalent of 32.18 ft/sec^2, or 9.8 meters/second^2. This is in close agreement with today's value.

[4] A teacher who apparently wanted to punish Gauss (a schoolboy of age 10) and his classmates for some misdeed told the class to add up the first 100 positive integers. Gauss, to the astonishment of the teacher, had the answer in seconds. Here is how he did it:

$$\begin{array}{rrrcrr} 1+ & 2+ & 3+ & \cdots + & 99+ & 100 \\ 100+ & 99+ & 98+ & \cdots + & 2+ & 1 \\ \hline 101+ & 101+ & 101+ & \cdots + & 101+ & 101 \end{array}$$

He had added the first 100 integers twice, in pairs, and gotten $101 \cdot 100 = 10,100$. So half this number, or 5,050, is the answer.

[5] In 1884 a conference of nations in Washington D.C. declared the meridian through Greenwich, England, to be the prime meridian, i.e., the line of 0° longitude. Greenwich had been the seat of an astronomical observatory since Newton's time. The time of day in Greenwich is known as Greenwich Mean Time (GMT).

[6] See the link A Computer Model of Elliptical Orbits (Generated with Kepler's Equations) on the web site http://www.nd.edu:80/ hahn/part1.html

[7] Comets contain matter left over from the formation of the solar system. They are studied by astronomers, among other reasons, for the information that they reveal about this formation process.

5

The Calculus of Leibniz

The factual understanding of our physical reality necessarily involves the clarification of numerical relationships. For example, Galileo's efforts to understand the path of a projectile require answers to questions such as: How high will it go? How far away will it strike the ground? What will its velocity be at the point of impact? Can its parabolic trajectory be calculated? An initial response to Galileo's questions is this: Information about the tangent lines to the path gives information about the path itself. Figure 5.1 illustrates this. For instance, the tangent at the initial point A provides the initial direction of the projectile, and the tangent at the terminal point C gives the angle of impact. At the point B, where the tangent is horizontal, the projectile reaches its highest elevation. The first concern of calculus is the study of tangent lines and the information that they provide about the curve.

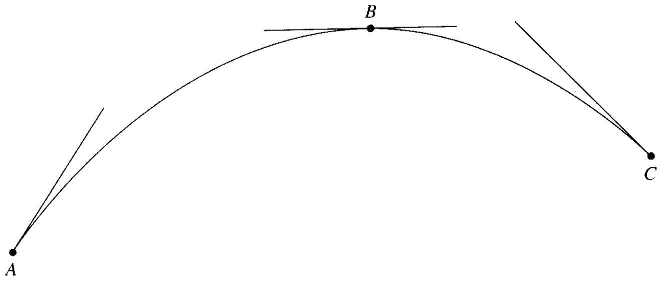


Figure 5.1

The second concern of calculus is the analysis of large sums of small numbers. An example is Archimedes's procedure for computing the area of the parabolic section by adding the areas of more and more triangles that are progressively smaller and smaller. (Refer to Section 3.4.) The principle of Cavalieri (see Section 4.6) is based, at least implicitly, on similar considerations. Consider a curve from A to B and a line segment L (as shown in Figure 5.2). The area between the curve and the segment can be computed to any degree of accuracy as follows: Place rectangles as shown and notice that their areas, when added together, approximate the area in question. To achieve a very high degree of accuracy, just take a huge number of such rectangles. Since each rectangle is very thin, its area is some very small number. Therefore, the area between the curve and the segment can be computed by an addition of lots of numbers, all very small.

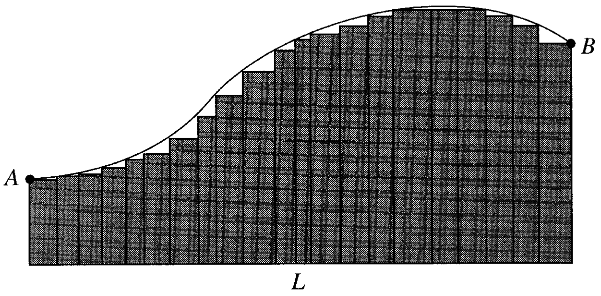


Figure 5.2

Now comes a surprise. The two basic concerns of calculus, as they have just been described, are closely related to each other. Namely, the relationship between a curve and its tangent line provides an effective strategy for the addition of a large number of very small terms.

Recall at this point the important lesson of analytic geometry, that numerical relationships can be interpreted and studied as graphs and hence in the context of curves. To understand the importance of this connection to the current discussion, consider the following: The calculation of the greatest tension in the cable of a suspension bridge or the maximum pressure generated by the explosion that propels a shell in a gun barrel both reduce to the determination of the highest point on a curve. And the ability to gain certain quantitative insights into the nature of gravity, or the energy generated by a force, is the ability to add the areas of a large array of very small numbers.

The points just made can be combined and summarized as follows. The domain of calculus is the quantitative study of certain properties of curves. It is therefore a discipline that concerns itself with the analysis of basic numerical relationships and consequently with quantitative aspects of much of the physical world around us.

Leibniz and Newton succeeded in creating calculus by building their insights, as well as those of their contemporaries and predecessors, into a general theory of curves and applications of that theory. This chapter will begin with a focus on the work of Leibniz.[1]

Gottfried Wilhelm Leibniz (1646–1716) was born in Leipzig, in the German state of Saxony. After obtaining a doctor of laws degree from the University of Nuremberg in 1667, he accepted a diplomatic post. In 1672 he was sent on a mission to Paris. Its purpose was to deflect the growing threat of France to the German states by attempting to persuade the government of Louis XIV to occupy Egypt. The rationale was that this would allow a strengthened, united Christian Europe to control the Eastern trade routes and repel the attacks of the Turkish Empire against Eastern Europe. This scheme failed, and a French attack on the German states soon followed. Leibniz's stay in Paris proved to be most fortuitous for mathematics, however.

In Paris, Leibniz came into contact with a number of brilliant scholars. A brief diplomatic journey early in 1673 brought him to London, where he met leading English scientists. The death of his patron ended his career as a diplomat, but he remained in Paris and resumed his earlier interests in mathematics with great intensity. Under the guidance of Christiaan Huygens (1629–1695), the Dutch mathematician and physicist, he studied Huygens's analysis of the pendulum, the *Géométrie* of Descartes, the work of Isaac Barrow (1630–1677), who was Newton's professor at Cambridge, and the *Geometria Indivisibilibus* and *Exercitationes Sex*[2] of Cavalieri. Leibniz mastered all of this, and more, very quickly. In the period from 1673 to 1676, he laid the foundations of calculus.

Unable to secure permanent employment in Paris, Leibniz reluctantly accepted the post of librarian and councilor to a German duke. In the service of the duke and his successors, he undertook a remarkable variety of tasks. He worked on a hydraulic press, on gears and steering devices for carriages, on water pumps driven by windmills, and he proposed a process for the desalinization of water. He speculated about the origins and the age of the Earth. His *Meditationes de Cognitione, Veritate, et Ideis* (Reflections on Knowledge, Truth, and Ideas) and *Discours de Métaphysique* (Discourse on Metaphysics) developed his philosophy. For Leibniz the world was infinite from both the qualitative and quantitative perspective; the number of substantial units, or *monads*, in the smallest particle was as boundless as the entire universe. Summarizing his philosophical thinking, he wrote, "It is as if God had wanted to create the universe in as many different ways as there are souls, or as if he created as many microcosms which are identical in their deepest foundations, but which are manifold in their appearance. This in a few words is my entire philosophy."

5.1 Straight Lines

Before considering Leibniz's mathematics, we first turn to the quantitative analysis of lines. Lines are automatically assumed to be straight. A line that is not straight is called a curve. We will be working with a plane that is equipped with an x-y coordinate system.

Consider the equation $2x - 3y - 4 = 0$. It can be rewritten in many equivalent ways, for example, $2x - 3y = 4$, or $3y = 2x - 4$, or $y = \frac{2}{3}x - \frac{4}{3}$. If the values $x = a$ and $y = b$ satisfy one of these equations, they satisfy all of them. Therefore, all of these equations have the same graph. We will work with

$$y = \frac{2}{3}x - \frac{4}{3}.$$

If $x = 5$, then $y = \frac{10}{3} - \frac{4}{3} = \frac{6}{3} = 2$. So $x = 5$ and $y = 2$ satisfy the equation, and the point $(5, 2)$ is on the graph. In the same way, $(4, \frac{4}{3})$, $(3, \frac{2}{3})$, $(2, 0)$, $(1, -\frac{2}{3})$, $(0, -\frac{4}{3})$, $(-2, -\frac{8}{3})$, and so on, are all on the graph. Plot these points, and observe that all fall on the single line pictured in Figure 5.3. Call this line L. Move along L from left to right and notice—compare the points

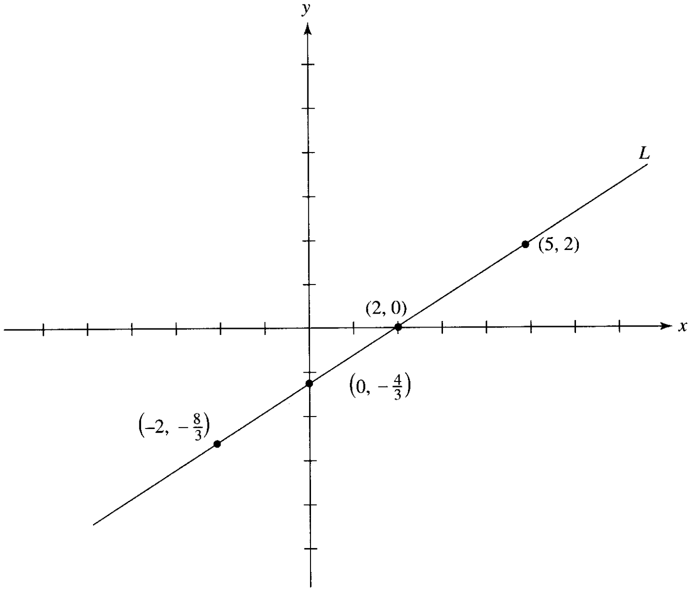


Figure 5.3

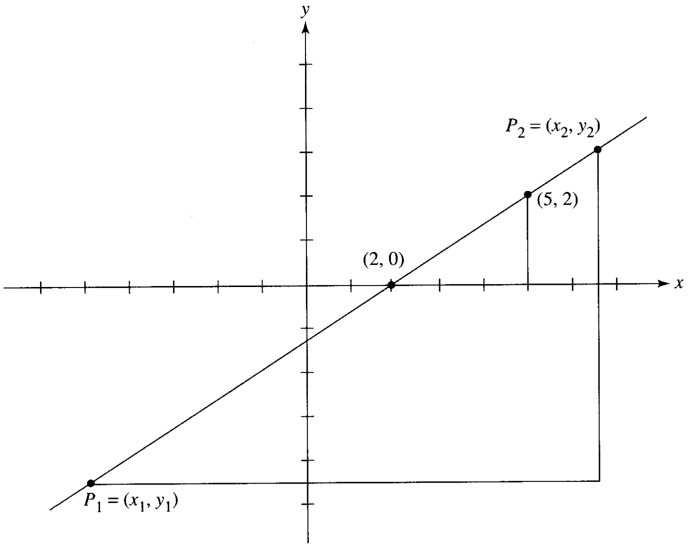


Figure 5.4

(2, 0) and (5, 2) for example—that for every 3 units of increase in the direction of the positive x axis, there is a corresponding increase of 2 units in the direction of the positive y-axis. So there is an increase of $\frac{2}{3}$ units in the y-coordinate for every increase of 1 unit in the x-coordinate. So the number $\frac{2}{3}$ measures the pitch, or steepness, of the line. It is called the *slope* of the line and is often denoted by m. In the case under consideration now, $m = \frac{2}{3}$.

The slope m can be determined as follows: Consider any two points on L, say, $(5,2)$ and $(2,0)$; form the difference $2 - 0 = 2$ between the y-coordinate of the first point and the y-coordinate of the second; form the difference $5 - 2 = 3$ between the x-coordinate of the first point and the x-coordinate of the second; and let $m = \frac{2}{3}$ be the ratio of the two differences. Any two distinct points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ on the line can be used to compute the slope. This is a consequence of the basic property of similar triangles. Refer to Figure 5.4 and notice that the right triangle with hypotenuse given by $(5,2)$ and $(2,0)$ is similar to the right triangle with hypotenuse P_1P_2. It follows that the ratios of the lengths of the vertical over the horizontal legs of the two triangles are equal. So 2 is to 3 as $y_2 - y_1$ is to $x_2 - x_1$, or

$$m = \frac{2}{3} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Again, the slope of a line is the ratio of the change in y over the corresponding change in x.

It should be clear that two nonvertical lines are parallel precisely if they have the same slope. Figure 5.5 shows several lines and their slopes. Notice that a line with a positive slope slants upward from left to right and that a line with a negative slope slants downward, again from left to right. Notice also that the steepest lines are the ones for which the absolute value of the slope is largest, and that a horizontal line has slope zero. Consider a line with slope $m \neq 0$. It is not very hard to verify that the line perpendicular to it has slope $-\frac{1}{m}$. Which pairs of lines in Figure 5.5 are perpendicular?

Example 5.1. Sketch the graph of the equation $5x + 3y = 15$ and determine its slope.

Solution. It suffices to find two points on the line and to draw the line through both. It is easiest to find the intercepts. Substituting $y = 0$ into the equation, we get $5x = 15$. So $x = 3$, and $(3,0)$ is on the graph.

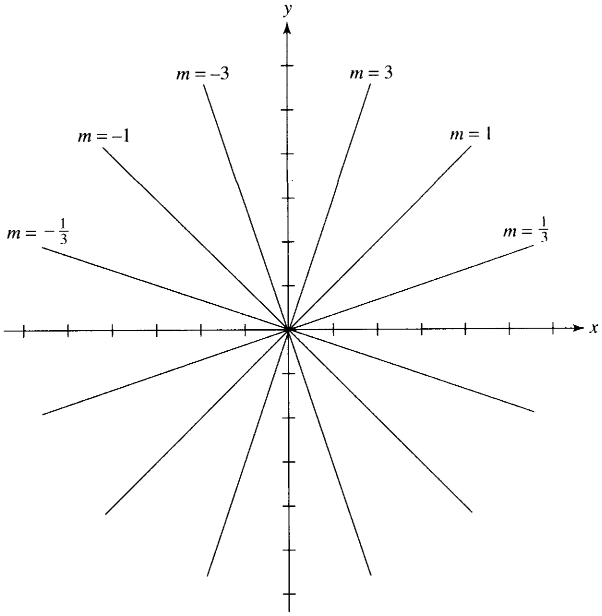


Figure 5.5

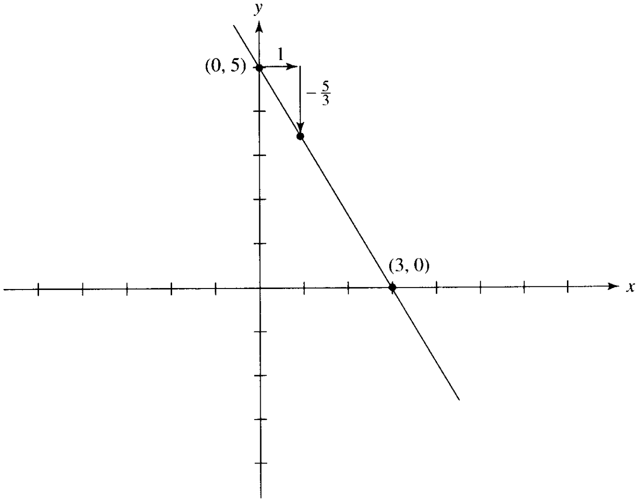


Figure 5.6

Substituting $x = 0$, we see that $(0,5)$ is on the graph. The graph is shown in Figure 5.6. Using these two points to compute the slope, we get

$$m = \frac{(5-0)}{(0-3)} = -\frac{5}{3}$$

So a change of $+3$ in x results in a change of -5 in y. Equivalently, a change of 1 unit in x produces a change of $-\frac{5}{3}$ units in y. Again, see Figure 5.6.

Consider the equation $2x+10 = 0$ or, equivalently, $x = -5$. The points (x, y) that satisfy it are precisely the points $(-5, y)$, where y can be anything. Notice, therefore, that the graph of the equation $x = -5$ is the vertical line through $(-5, 0)$. For any two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ on this line, $x_1 = x_2 = 5$, so the ratio $\frac{y_2-y_1}{x_2-x_1}$ is meaningless. This line has no slope. For the same reason, no vertical line has a slope.

Suppose that L is a nonvertical line. Select any two distinct points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ on L. As before, its slope is $m = \frac{y_2-y_1}{x_2-x_1}$. A look at Figure 5.7 shows that a point $P = (x, y)$ is on L precisely if the segment P_1P is parallel to L. So $P = (x, y)$ is on L precisely if the slope of the segment P_1P is equal to m. Since the slope of P_1P is $\frac{y-y_1}{x-x_1}$, the point $P = (x, y)$ is on L precisely if $\frac{y-y_1}{x-x_1} = m$. This equation can be rewritten in the form

$$y - y_1 = m(x - x_1).$$

Observe that it is also satisfied when $x = x_1$ and $y = y_1$. Therefore, it is an equation of the given line L. An equation of a line arranged in this way is said to be in *point-slope* form.

Example 5.2. Find an equation of the line through $(1, -7)$ with slope $-\frac{1}{2}$.

Solution. Using the point-slope form of the equation with $m = -\frac{1}{2}$, $x_1 = 1$, and $y_1 = -7$, we get

$$y + 7 = -\frac{1}{2}(x - 1).$$

This, as well as $2y + 14 = -x + 1$ or $x + 2y + 13 = 0$, are equations of the given line.

Example 5.3. Find an equation of the line through the points $(-1, 2)$, and $(3, -4)$.

Solution. The slope of the line is

$$m = \frac{-4-2}{3-(-1)} = -\frac{3}{2}.$$

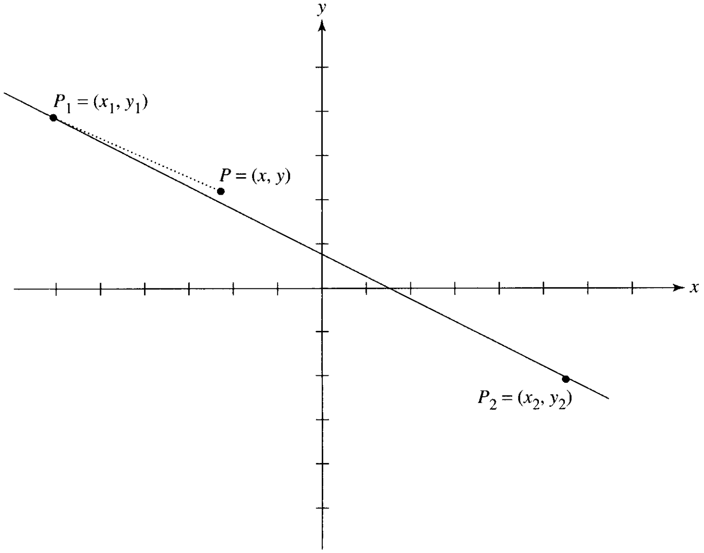


Figure 5.7

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$y - 2 = -\frac{3}{2}(x + 1).$$

This equation can be rewritten as $3x + 2y = 1$.

Suppose a nonvertical line has slope m and y-intercept b. This means that it intersects the y-axis at the point $(0, b)$. Taking $x_1 = 0$ and $y_1 = b$, the point-slope form of the equation of the line is $y - b = m(x - 0)$. Therefore, an equation of the line with slope m and y-intercept b is

$$y = mx + b.$$

An equation of a line arranged in this way is said to be in *slope-intercept* form. If a line is horizontal, its slope is $m = 0$. So the equation of such a line has the form $y = b$, where b is the y-intercept.

Take any equation of the form $Ax + By + C = 0$, where A, B, and C are constants. If $A = B = 0$, then $C = 0$. Disregard this case, and suppose that either $A \neq 0$ or $B \neq 0$. Suppose that $B \neq 0$. Then $Ax + By + C = 0$ is equivalent to $y = -\frac{A}{B}x - \frac{C}{B}$. A look at the point-slope form shows that this is an equation of the line with slope $-\frac{A}{B}$ and y-intercept $-\frac{C}{B}$. If $B = 0$, then $A \neq 0$. Now the equation is $Ax + C = 0$ or, equivalently, $x = -\frac{C}{A}$. As in the case $x = -5$, this is an equation of the vertical line through $(-\frac{C}{A}, 0)$. Therefore, the graph of any equation of the form $Ax+By+C$ is a line (unless $A = B = C = 0$). Any such equation is therefore called a *linear* equation.

Lines are important in the analysis of the hyperbola, the one conic section we have not as yet had a closer look at. (Refer to Sections 4.3 and 4.5.) Take any hyperbola and move it in such a way that its focal points (see Section 3.1) lie on the x-axis with the origin at the midpoint of the segment that joins the focal points. Doing what was done in Section 4.5 for the ellipse, it can be verified that such a hyperbola has an equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

with $a > 0$ and $b > 0$. To find the x-intercepts, set $y = 0$ and obtain $x^2 = a^2$ and $x = \pm a$. If $x = 0$, then $y^2 = -b^2$. But this is impossible, so there is no y-intercept. In fact, since

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1,$$

$x^2 \geq a^2$ and so $|x| = \sqrt{x^2} \geq a$. Therefore, $x \geq a$ or $x \leq -a$. This means that the hyperbola consists of two separate parts, one to the left of the line $x = -a$ and the other to the right of the line $x = a$. What else is going on?

Since $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, it follows that $\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1 = \frac{x^2-a^2}{a^2}$. So $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$ and, taking square roots, $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$. Observe that if x^2 is large relative to a^2, then the effect of a^2 can be discounted and $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$ is approximately equal to $\pm\frac{b}{a}\sqrt{x^2} = \pm\frac{b}{a}|x|$. Consider a point (x, y) on the hyperbola with x large and positive. If $y > 0$, then, by the observation just made, $y \approx \frac{b}{a}x$, so that the point is close to the line $y = \frac{b}{a}x$. If $y < 0$, then in the same way, the point is close to the line $y = -\frac{b}{a}x$. This analysis can be continued and refined to show that the graph of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has the general form shown in Figure 5.8. Notice that if a point (x, y) is on the portion of the curve above the x-axis, then $y \geq 0$, so that $y = \frac{b}{a}\sqrt{x^2 - a^2}$. If (x, y) lies below that x-axis, then $y = -\frac{b}{a}\sqrt{x^2 - a^2}$.

By interchanging the roles of x and y we get the equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

This also represents a hyperbola. Its graph is obtained by rotating that of Figure 5.8 by 90°.

Example 5.4. Sketch the curve $4x^2 - 4y^2 = 8$.

Solution. Dividing both sides by 8, we obtain

$$\frac{x^2}{(\sqrt{2})^2} - \frac{y^2}{(\sqrt{2})^2} = 1.$$

Letting $a = \sqrt{2}$ and $b = \sqrt{2}$, we see that this equation has the form discussed earlier. So its graph

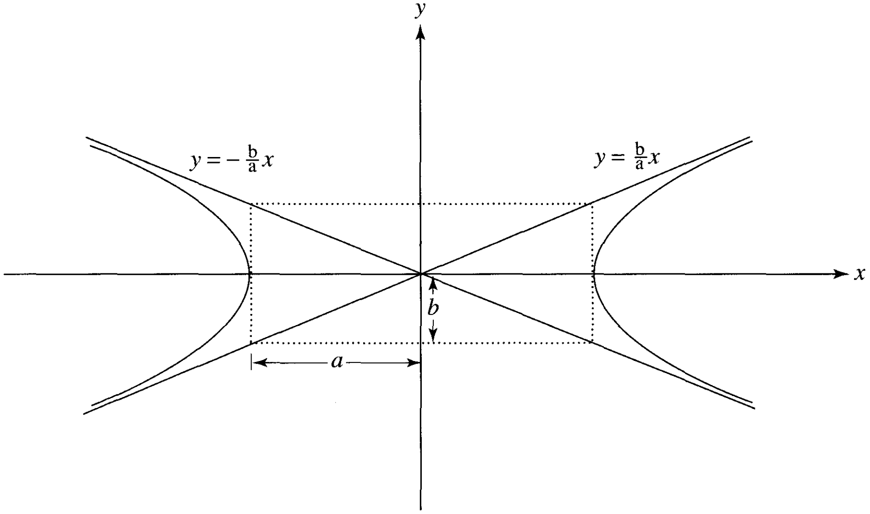


Figure 5.8

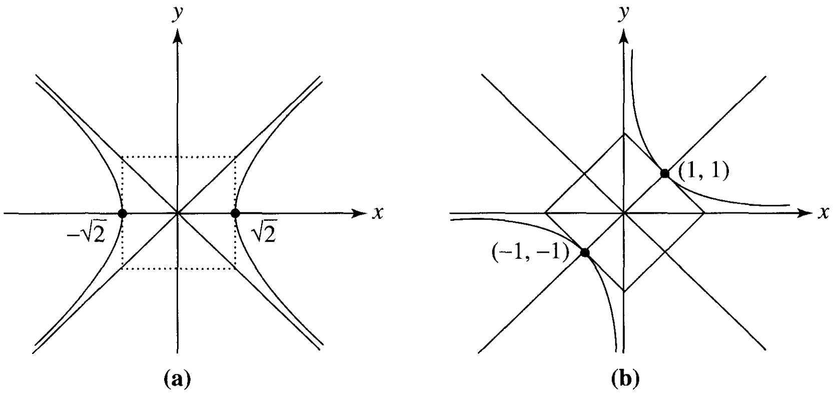


Figure 5.9

is the hyperbola obtained by taking $a = b = \sqrt{2}$ in Figure 5.8. It is sketched in Figure 5.9(a). Rotating this hyperbola by 45° gives the hyperbola in Figure 5.9(b). It is the graph of the equation $xy = 1$.

An equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, C, D, E, and F are constants with at least one of A, B, or C not zero, is called a *quadratic equation*. Observe that the hyperbolas of this section, the parabolas of Section 4.3, and the ellipses of Section 4.5 are all given by quadratic equations. Proceeding in the other direction, it can be shown that the graph of a quadratic equation is either a parabola, an ellipse, a hyperbola (all rotated and shifted in any possible way), or a pair of lines (for example, $xy = 0$), or a point (for example, $x^2 + y^2 = 0$), or it has no points at all (for example, $x^2 + 1 = 0$).

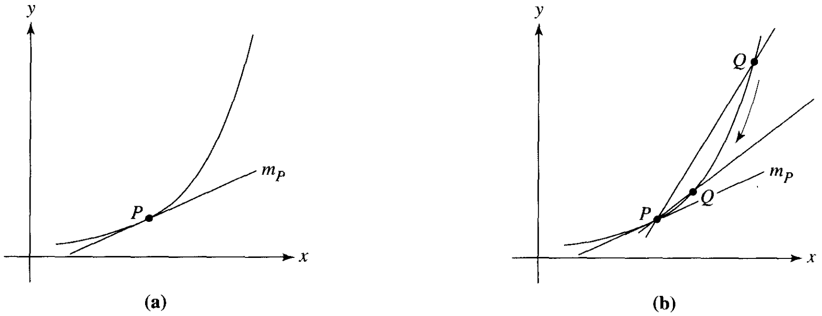


Figure 5.10

Now that we know a few things about lines, we can turn to tangent lines and the work of Leibniz.

5.2 Tangent Lines to Curves

Leibniz's first publication of his work on calculus is the article *Nova methodus pro maximis et minimis, itemque tangentibus … calculi genus* ("A new method for maxima and minima as well as tangents, which is impeded neither by fractional nor by irrational quantities, and a remarkable type of calculus for this"), which appeared in 1684. It presents a new notation, introduces the term "calculus differentialis," contains the general rules of differentiation … and is nearly impossible to read. Even Leibniz's friends, the brothers Bernoulli (famous mathematicians themselves), commented that this article is "an enigma rather than an explication." We advisedly turn instead to a later manuscript. In it, Leibniz provides details and corrections of his earlier exposition. The discussion of the work of Leibniz that follows in this and subsequent sections is a much-expanded version of his original manuscripts. It also adds concepts and notational elements that were not introduced until later. These serve to clarify some of the remaining "enigmas." However, the essence of Leibniz's contributions are retained.

Let a curve in the x-y plane be given. Regard it as the graph of some equation in x and y. Fix a point P on the curve and consider the tangent line to the curve at P. Assume that it is not vertical and let m_P be its slope. See Figure 5.10(a). Now take some other point $Q \neq P$ on the curve. While keeping P fixed, push Q towards P. Refer to Figure 5.10(b) and notice that as Q closes in on P, the line through P and Q closes in on the tangent line at P. So as Q is pushed to P, the slope of the line through P and Q homes in on the slope m_P of the tangent.

Let's have a more algebraic look at what is going on. Let x and y be the coordinates of P and let x' and y' be the coordinates of Q. Let $\Delta x = x' - x$ and $\Delta y = y' - y$ be the differences between the x-coordinates and y-coordinates of P and Q. So the coordinates of Q are $x' = x + \Delta x$ and $y' = y + \Delta y$. Again draw the line through P and Q and observe that the slope of this line is equal to the ratio

$$\frac{(y + \Delta y) - y}{(x + \Delta x) - x} = \frac{\Delta y}{\Delta x}.$$

Again, keeping P fixed push Q towards P. Observe from Figure 5.11 that pushing Q to P is the same thing as pushing Δx to zero. Therefore, when Δx is pushed to zero, the slope $\frac{\Delta y}{\Delta x}$ of the line through P and Q closes in on the slope of the tangent. Rewritten in limit notation, what we have observed is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = m_P.$$

Notice that Leibniz cannot simply set $\Delta x = 0$, for then (see Figure 5.11) Δy is also equal to zero. Thus

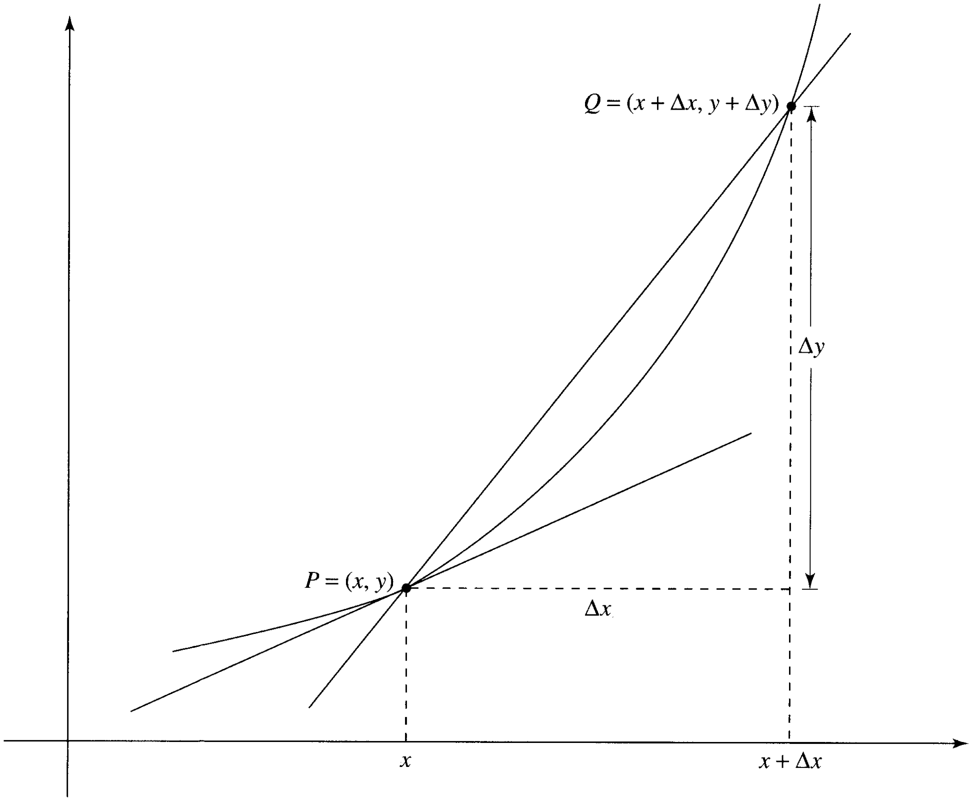


Figure 5.11

$\frac{\Delta y}{\Delta x}$ would be $\frac{0}{0}$, which is not defined, i.e., it does not make sense.

Leibniz turns to the parabola to illustrate his method. He takes $y = Ax^2$, where A is any constant. Let's be more specific and consider $y = x^2$. Instead of a general point $P = (x, y)$ on the parabola, let's take $P = (3, 9)$. Now $Q = (3 + \Delta x, 9 + \Delta y)$. Since Q is on the graph,

$$9 + \Delta y = (3 + \Delta x)^2 = 3^2 + 2 \cdot 3\Delta x + (\Delta x)^2.$$

Subtracting 9 from both sides gives us $\Delta y = 6\Delta x + (\Delta x)^2$, and after dividing both sides by Δx, we get

$$\frac{\Delta y}{\Delta x} = 6 + \Delta x.$$

By pushing Δx to zero,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 6.$$

Therefore, the slope of the tangent line to the graph of $y = x^2$ at the point $P = (3, 9)$ is 6. So in this case $m_P = 6$. See Figure 5.12.

Now let $P = (x, y)$ be any point on the parabola $y = x^2$. Since

$$Q = (x + \Delta x, y + \Delta y)$$

is on the graph, $y + \Delta y = (x + \Delta x)^2$. We continue in Leibniz's own words (in English translation):

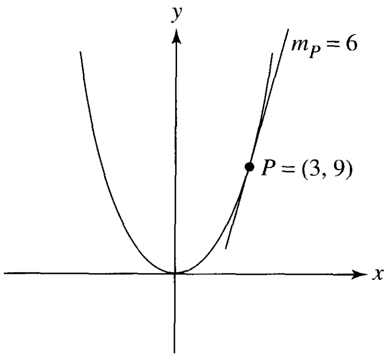


Figure 5.12

Then, since $y = x^2$, by the same law, we have

$$y + \Delta y = (x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2;$$

and taking away the y from the one side and the x^2 from the other, we have left

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x.$$

Make use of modern limit notation and observe that Leibniz can now conclude that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x.$$

So the slope of the tangent line at any point $P = (x, y)$ on the parabola $y = x^2$ is $m_P = 2x$. Taking $x = 3$, we get the point $P = (3, 9)$ and $m_P = 6$ (as we already saw).

Example 5.5. Find the slope m_P of the tangent line of the parabola $y = x^2$ at the point $P = (-4, 16)$.

Solution. The slope of the tangent at any point (x, y) is $2x$ for any x. Since $x = -4$, it follows that $m_P = 2 \cdot (-4) = -8$ at $P = (-4, 16)$.

Leibniz next turns to the graph of the equation $y = Ax^3$ where A is a constant. As before, $P = (x, y)$ is a fixed point and $Q = (x + \Delta x, y + \Delta y)$ any other point on the graph. Another look at Figure 5.11 will let you visualize what is going on. Since both P and Q are on the graph, $y = Ax^3$ and

$$\begin{aligned} y + \Delta y &= A(x + \Delta x)^3 \\ &= A\left(x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3\right). \end{aligned}$$

After subtracting $y = Ax^3$ from both sides, he gets $\Delta y = 3Ax^2\Delta x + 3Ax(\Delta x)^2 + A(\Delta x)^3$. Dividing both sides by Δx, he has

$$\frac{\Delta y}{\Delta x} = 3Ax^2 + 3xA\Delta x + A(\Delta x)^2.$$

Pushing Δx to zero, Leibniz can conclude that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left(3Ax^2 + 3xA\Delta x + A(\Delta x)^2\right) = 3Ax^2.$$

He has shown that the slope of the tangent line to the curve $y = Ax^3$ at any point $P = (x, y)$ is equal to $m_P = 3Ax^2$.

Example 5.6. Consider the curve $y = x^3$ and the point $P = (2, 8)$ on it. Taking $A = 1$ and $x = 2$ in the discussion just completed, we see that the slope of the tangent at P is $m_P = 3 \cdot 2^2 = 12$. Notice that the equation of the tangent is $y - 8 = 12(x - 2)$, or $y = 12x - 16$.

What happens when Leibniz's method is applied to a line? Let's take $y = \frac{1}{2}x + 1$. See Figure 5.13. Since P and Q both satisfy this equation, we get

$$y + \Delta y = \frac{1}{2}(x + \Delta x) + 1 = \frac{1}{2}\Delta x + \frac{1}{2}x + 1,$$

and hence $\Delta y = \frac{1}{2}\Delta x$. So $\frac{\Delta y}{\Delta x} = \frac{1}{2}$. Pushing Δx to zero has no effect on $\frac{1}{2}$, and therefore $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{2}$. This confirms the obvious: The tangent at any point P on the line is the line itself, and it has slope $\frac{1}{2}$.

Consider the parabola $x = y^2$ next. As before, fix a point $P = (x, y)$ on the parabola and let $Q = (x + \Delta x, y + \Delta y)$ be any other point on it. Since the coordinates of Q satisfy the equation of the parabola,

$$x + \Delta x = (y + \Delta y)^2 = y^2 + 2y\Delta y + (\Delta y)^2.$$

Since P does also,

$$\Delta x = 2y\Delta y + (\Delta y)^2 = \Delta y(2y + \Delta y)$$

by subtracting and factoring. Recall that the focus is on $\frac{\Delta y}{\Delta x} = \frac{1}{2y+\Delta y}$. Push Δx to zero and refer to Figure 5.11. As Q approaches P, the quantity Δy goes to zero. It follows, therefore, that the slope of the tangent to the parabola at $P = (x, y)$ is

$$m_P = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{2y}.$$

Example 5.7. What is the slope of the tangent of the parabola $x = y^2$ at $P = (3, \sqrt{3})$? What is the slope of the tangent at the origin?

Solution. Since $y = \sqrt{3}$ it follows from the formula just established that $m_P = \frac{1}{2\sqrt{3}}$. The origin is the point $P = (0, 0)$. When $y = 0$, the formula $m_P = \frac{1}{2y}$ does not make sense. This is explained by the fact that the tangent to the parabola $x = y^2$ at the origin is vertical. So this tangent has no slope. Confirm this by sketching a graph of $x = y^2$.

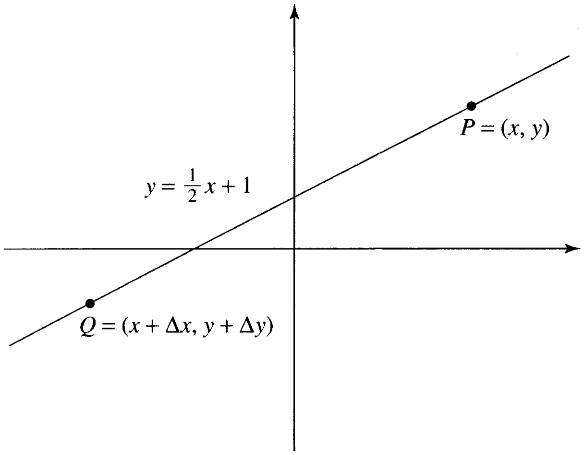


Figure 5.13

Finally, consider the circle $x^2 + y^2 = 5^2$. Once more, let $P = (x, y)$ be a fixed point on the circle and let $Q = (x + \Delta x, y + \Delta y)$ be another point on the circle. Since Q is on the circle, $(x + \Delta x)^2 + (y + \Delta y)^2 = 5^2$, so

$$x^2 + 2x\Delta x + (\Delta x)^2 + y^2 + 2y\Delta y + (\Delta y)^2 = 25.$$

Since $x^2 + y^2 = 5^2$, we get that

$$2x\Delta x + (\Delta x)^2 + 2y\Delta y + (\Delta y)^2 = 0.$$

After collecting the Δy terms on the left and the Δx terms on the right and factoring, we see that

$$\Delta y(2y + \Delta y) = -\Delta x(2x + \Delta x).$$

We need to concentrate on the ratio

$$\frac{\Delta y}{\Delta x} = -\frac{2x + \Delta x}{2y + \Delta y}.$$

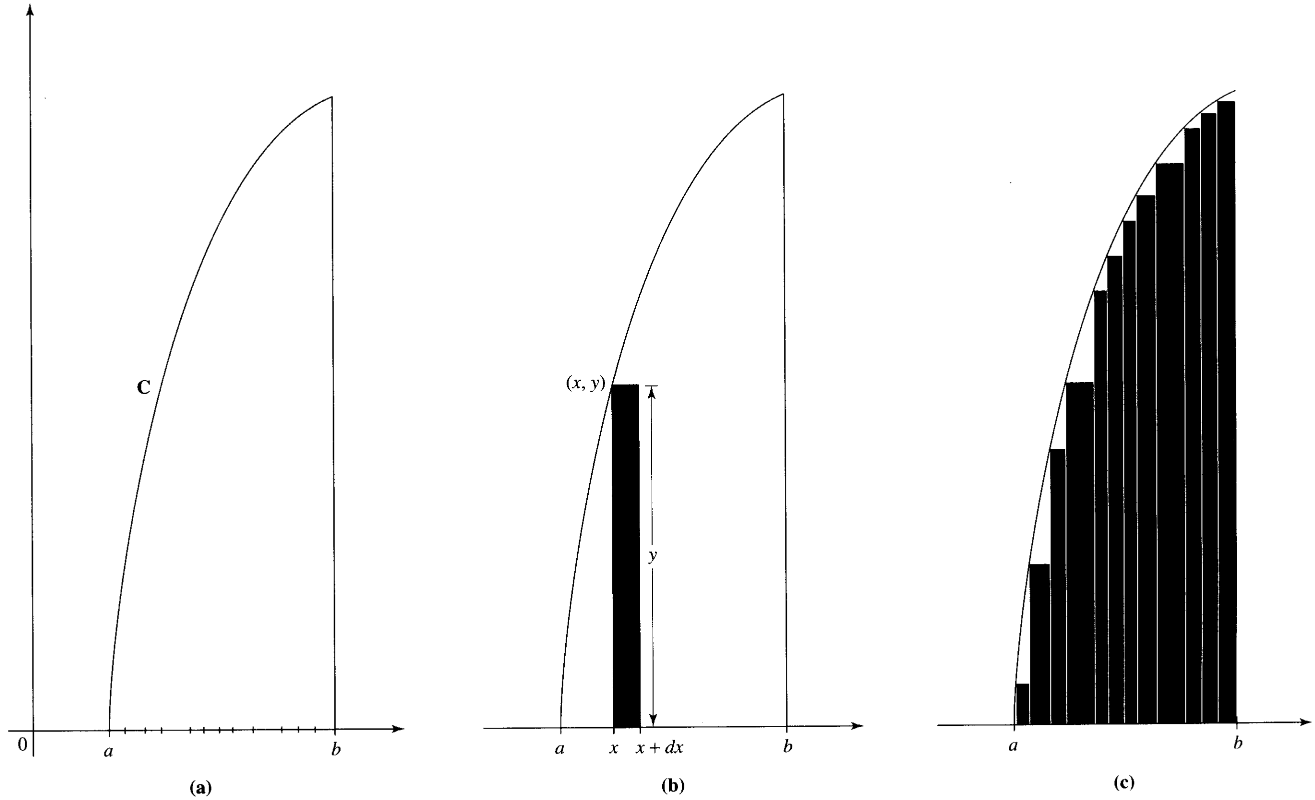
Now push Δx to zero. Refer back to Figure 5.11 once more. Since Q goes to P, note that Δy goes to zero. So it follows that the slope of the tangent to the circle at the point $P = (x, y)$ is

$$m_P = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\frac{2x}{2y} = -\frac{x}{y}.$$

Example 5.8. Observe that the slope of the tangent to the circle $x^2 + y^2 = 5^2$ at the point $P = (3, 4)$ is $-\frac{3}{4}$. What is the problem at the points $(5, 0)$ and $(-5, 0)$?

5.3 Areas and Differentials

In a manuscript written in October 1675, Leibniz considers a coordinate system and a curve **C**, as shown in Figure 5.14(a). He places a few points on the x-axis between a and b, starting with a and ending with b. He denotes a typical one of these points by x and lets dx be the distance between it and the next point. Consider any two successive points that Leibniz has placed on the axis. Let x be the first one; notice that the second one is $x + dx$. Let y be the y-coordinate of the point on **C** that lies above x. Draw the rectangle determined by x and $x + dx$, along with the point (x, y). This rectangle is shown in Figure 5.14(b). Observe that its area is $y \cdot dx$. Figure 5.14(c) shows all of the rectangles that are obtained in this way from the points that Leibniz has selected. Observe that these rectangles, when taken together, fill out—in an approximate way—the region under the curve **C** over the interval from a to b.



(a) (b) (c)

Figure 5.14

Let's have a look at some very specific examples of what Leibniz had in mind.

Example 5.9. Consider the area under the line $y = \frac{2}{3}x + 1$ and over the interval from 1 to 5. Insert the points

$$1 \leq 1.7 \leq 2.6 \leq 3.3 \leq 3.8 \leq 4.4 \leq 5$$

between 1 and 5. The first point is $x = 1$, so the corresponding y is $y = \frac{2}{3} \cdot 1 + 1 = \frac{5}{3}$; the first $dx = 1.7 - 1 = 0.7 = \frac{7}{10}$, and the first $y \cdot dx = \frac{5}{3} \cdot \frac{7}{10} = \frac{35}{30} = \frac{175}{150}$. The second point is $x = 1.7 = \frac{17}{10}$, so the corresponding y is $y = \frac{2}{3} \cdot \frac{17}{10} + 1 = \frac{32}{15}$; the second $dx = 2.6 - 1.7 = 0.9 = \frac{9}{10}$, and the second $y \cdot dx = \frac{32}{15} \cdot \frac{9}{10} = \frac{288}{150}$. The third point is $x = 2.6 = \frac{26}{10}$, so the corresponding y is $y = \frac{2}{3} \cdot \frac{26}{10} + 1 = \frac{41}{15}$; the third $dx = 3.3 - 2.6 = 0.7 = \frac{7}{10}$, and the third $y \cdot dx = \frac{41}{15} \cdot \frac{7}{10} = \frac{287}{150}$. The fourth point is $x = 3.3 = \frac{33}{10}$, so the corresponding y is $y = \frac{2}{3} \cdot \frac{33}{10} + 1 = \frac{48}{15}$; the fourth $dx = 3.8 - 3.3 = 0.5 = \frac{5}{10}$, and the fourth $y \cdot dx = \frac{48}{15} \cdot \frac{5}{10} = \frac{240}{150}$. The fifth point is $x = 3.8 = \frac{38}{10}$, so the corresponding y is $y = \frac{2}{3} \cdot \frac{38}{10} + 1 = \frac{53}{15}$; the fifth $dx = 4.4 - 3.8 = 0.6 = \frac{6}{10}$, and the fifth $y \cdot dx = \frac{53}{15} \cdot \frac{6}{10} = \frac{318}{150}$. The sixth point is $x = 4.4 = \frac{44}{10}$, so the corresponding y is $y = \frac{2}{3} \cdot \frac{44}{10} + 1 = \frac{59}{15}$; the sixth $dx = 5 - 4.4 = 0.6 = \frac{6}{10}$, and the sixth $y \cdot dx = \frac{59}{15} \cdot \frac{6}{10} = \frac{354}{150}$. All the points have been dealt with and the process is complete. The sum of all the rectangular areas $y \cdot dx$ that the points $1 \leq 1.7 \leq 2.6 \leq 3.3 \leq 3.8 \leq 4.4 \leq 5$ determine is

$$\frac{175}{150} + \frac{288}{150} + \frac{287}{150} + \frac{240}{150} + \frac{318}{150} + \frac{354}{150} = \frac{1662}{150} = 11.08.$$

What is the actual area under the line between 1 and 5? If $x = -\frac{3}{2}$, then $y = \frac{2}{3}(-\frac{3}{2}) + 1 = -1 + 1 = 0$, and so the line $y = \frac{2}{3}x + 1$ crosses the x-axis at

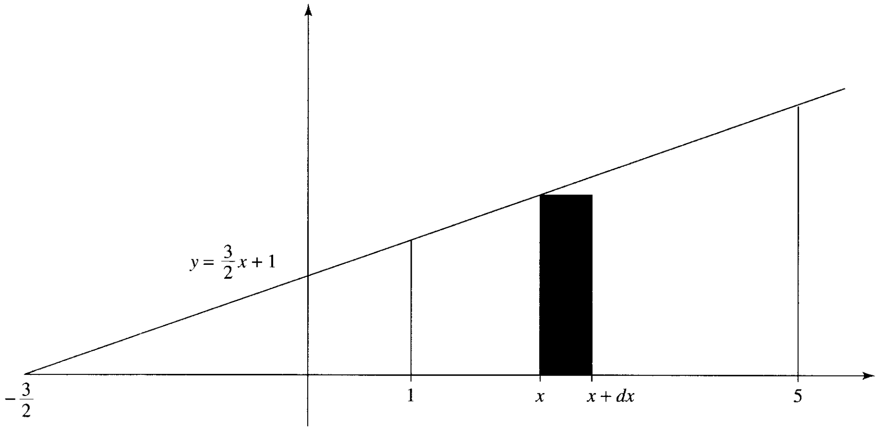


Figure 5.15

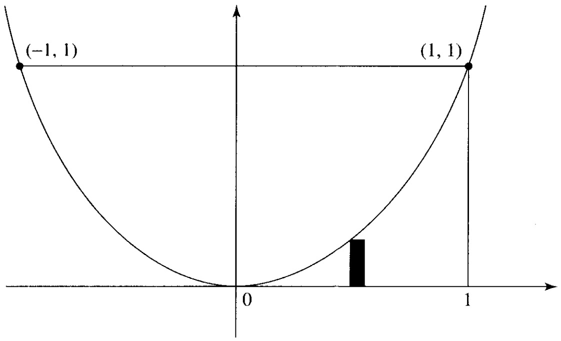


Figure 5.16

$x = -\frac{3}{2}$, as shown in Figure 5.15. So the actual area is the difference between two triangles: a larger triangle with base $5 + \frac{3}{2} = \frac{13}{2}$, height $\frac{2}{3} \cdot 5 + 1 = \frac{13}{3}$, and area $\frac{1}{2} \cdot \frac{13}{2} \cdot \frac{13}{3} = \frac{169}{12}$; and a smaller triangle with base $\frac{3}{2} + 1 = \frac{5}{2}$, height $\frac{2}{3} \cdot 1 + 1 = \frac{5}{3}$, and area $\frac{1}{2} \cdot \frac{5}{2} \cdot \frac{5}{3} = \frac{25}{12}$. The actual area is the difference, $\frac{169}{12} - \frac{25}{12} = \frac{144}{12} = 12$. So 11.08 is an approximation, but only a rough one.

Example 5.10. Next let's look at the parabola $y = x^2$ over the interval $0 \leq x \leq 1$, as pictured in Figure 5.16. Consider the points

$$0 \leq 0.1 \leq 0.3 \leq 0.5 \leq 0.8 \leq 1$$

between 0 and 1. The first point is $x = 0$, so the corresponding y is 0; the first dx is $0.1 - 0 = 0.1$, and the first $y \cdot dx = (0)(0.1) = 0$. The second point is $x = 0.1$, so the corresponding y is $(0.1)^2 = 0.01$; the second $dx = 0.2$, and $y \cdot dx = (0.01)(0.2) = 0.002$. The third point is $x = 0.3$, so the corresponding y is $(0.3)^2 = 0.09$; the third dx is 0.2, and $y \cdot dx = (0.09)(0.2) = 0.018$. The fourth point is $x = 0.5$, so the corresponding y is $(0.5)^2 = 0.25$; the fourth dx is 0.3, and $y \cdot dx = (0.25)(0.3) = 0.075$. The fifth point is $x = 0.8$, so the corresponding y is $(0.8)^2 = 0.64$; the fifth dx is $1 - 0.8 = 0.2$, and $y \cdot dx = (0.64)(0.2) = 0.128$. The sum of all the $y \cdot dx$ is equal to

$$0 + 0.002 + 0.018 + 0.075 + 0.128 = 0.223.$$

How good is this approximation of the area under the parabola from 0 to 1? The actual area can be computed as follows. The area of the triangle determined by the points $(-1, 1)$, $(1, 1)$, and the origin, has base 2 and height 1. So its area is 1 and hence by Archimedes's theorem, the area of the parabolic sector cut by the segment from $(-1, 1)$ to $(1, 1)$ is $\frac{4}{3} \cdot 1 = \frac{4}{3}$. It follows that the area *under* the parabola from -1 to 1 is $\frac{2}{3}$. So the area under the parabola from 0 to 1 is $\frac{1}{3} \approx 0.333$.

In Example 5.9, Leibniz's method provided the approximation 11.08 for an area that is actually 12. Since $\frac{12 - 11.08}{12} = \frac{0.92}{12} \approx 0.08$, this approximation is off by about 8%. In the case of Example 5.10, note that $\frac{0.333-0.223}{0.333} \approx \frac{0.11}{0.33} \approx 0.33$. So this error is a substantial 33%. Is there a flaw in Leibniz's method? No. The problem was simply that not enough rectangles were taken! For example, if many more points than just $0.1 \leq 0.3 \leq 0.5 \leq 0.8$ are inserted between 0 and 1, then much more satisfactory approximations can be achieved for the area of the parabolic region.

Return to Figure 5.14 and to Leibniz's discussion. Suppose now that there are lots of points between a and b on the x-axis and that they are all packed very tightly together. Think big here, as Archimedes did when he considered the universe packed with grains of sand. These points will give rise to a huge number of very thin rectangles under the curve. These rectangles, one next to the other, will together fill out the area under the curve **C**. Figure 5.17 illustrates this for the portion of the parabola that is considered in Example 5.10. Now, the sum of the areas of the rectangles will, for all practical purposes, be numerically equal to the area under the parabola, in the same way that, say, 0.33333333 is "equal" to $\frac{1}{3}$.

After various experiments with notation, Leibniz finally uses an elongated S (the Latin word for *Sum* is *Summa*) and writes $\int y\,dx$ to denote[3] the sum of all the $y \cdot dx$ determined by the very tightly packed set of points between a and b. In 1822, the French mathematician Fourier inserted a and b into the notation—to indicate that the terms run from $x = a$ to $x = b$. He thus wrote this sum as

$$\int_b^a y\,dx.$$

This is the modern notation, and we will make use of it here.

Example 5.11. Return to Figure 5.15 and the area under the line $y = \frac{2}{3}x + 1$ over the interval from 1 to 5. This time, insert the points

$$1 \leq 1.0001 \leq 1.0002 \leq 1.0003 \leq \ldots$$
$$\leq 4.9997 \leq 4.9998 \leq 4.9999 \leq 5$$

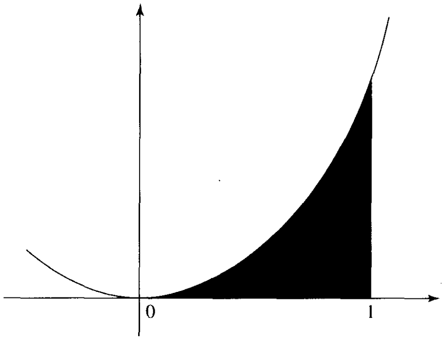


Figure 5.17

between 1 and 5. The first x is 1, the second is $1.0001 = 1 + \frac{1}{10{,}000}$, the third is $1.0002 = 1 + \frac{2}{10{,}000}$, the fourth is $x = 1.0003 = 1 + \frac{3}{10{,}000}, \ldots$, and the last is $x = 4.9999 = 1 + 3.9999 = 1 + \frac{39999}{10{,}000}$. Therefore, the typical x has the form $1 + \frac{i}{10{,}000}$ for i between 0 and 39,999. (Notice that tens of thousands of points have been packed between 1 and 5.) Since $y = \frac{2}{3}x + 1$, it follows that the typical $y \cdot dx$ is

$$y \cdot dx = \left[\frac{2}{3}\left(1 + \frac{i}{10{,}000}\right) + 1\right] \cdot \frac{1}{10{,}000}.$$

This can be rewritten as follows:

$$\begin{aligned} y \cdot dx &= \frac{1}{10{,}000} \cdot \left[\frac{2}{3}\left(1 + \frac{i}{10{,}000} + \frac{3}{2}\right)\right] \\ &= \frac{2}{30{,}000} \cdot \left(1 + \frac{i}{10{,}000} + \frac{3}{2}\right) \\ &= \frac{1}{15{,}000} \cdot \left(\frac{5}{2} + \frac{i}{10{,}000}\right) \\ &= \frac{1}{15{,}000} \cdot \left(\frac{25{,}000}{10{,}000} + \frac{i}{10{,}000}\right) \\ &= \frac{1}{1.5 \times 10^8} \cdot (25{,}000 + i). \end{aligned}$$

By substituting $i = 0, 1, 2, \ldots, 39{,}999$ into this expression and adding everything, we see that the sum of all the $y \cdot dx$ is equal to

$$\frac{1}{1.5 \times 10^8} \cdot (25{,}000 + 25{,}001 + 25{,}002 + \cdots + 64{,}999).$$

The terms inside the parentheses can be added by the technique of Gauss the schoolboy (see Note 4 at the end of Chapter 4), as follows:

$$\begin{array}{ccccccccccc}
25{,}000 & + & 25{,}001 & + & 25{,}002 & + & \cdots & + & 64{,}998 & + & 64{,}999 \\
64{,}999 & + & 64{,}998 & + & 64{,}997 & + & \cdots & + & 25{,}001 & + & 25{,}000 \\
\hline
89{,}999 & + & 89{,}999 & + & 89{,}999 & + & \cdots & + & 89{,}999 & + & 89{,}999
\end{array}$$

Recall that i ranges from 0 to 39,999; therefore, each line in the preceding addition has 40,000 terms. It follows that twice

$$25{,}000 + 25{,}001 + 25{,}002 + \cdots + 64{,}999$$

is equal to $(40{,}000)(89{,}999) = 35.9996 \times 10^8$. So

$$25{,}000+25{,}001+25{,}002+\cdots+64{,}999 = 17.9998 \times 10^8.$$

Inserting this into the earlier expression for the sum of all the $y \cdot dx$, we get

$$\frac{1}{1.5 \times 10^8} \cdot \left(17.9998 \times 10^8\right) = \frac{17.9998}{1.5} = 11.9999$$

with four-decimal accuracy. Refer back to Example 5.9 for the fact that the area under the line $y = \frac{2}{3}x+1$ over the interval from 1 to 5 is 12. So we now have what we wanted: A good approximation of the area under the curve. The equation

$$\int_1^5 y\,dx = \int_1^5 \left(\frac{2}{3}x+1\right) dx = 11.9999 = 12$$

summarizes what was done in Leibniz's notation.

It is important to think of the symbol $\int_a^b y\,dx$ as more than just a number equal to the area under the curve. Include in your thinking the summation process—of a very large number of very small terms—that produces this number. This way of thinking about area will soon have important consequences.

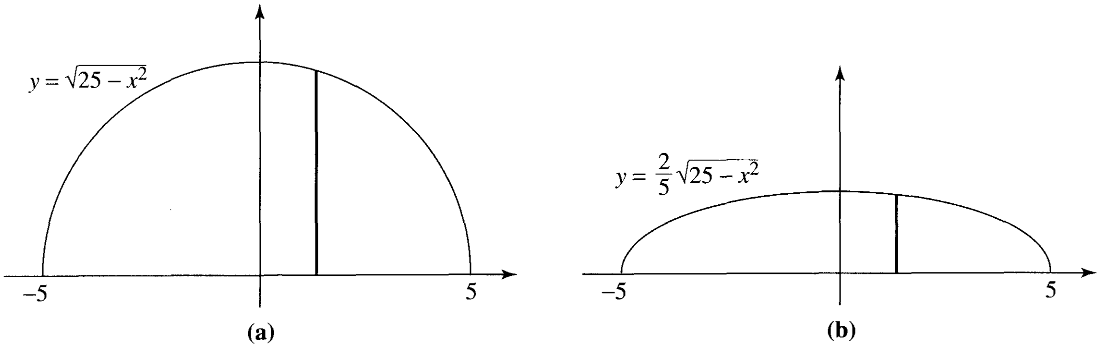
The formula

$$(*) \qquad \boxed{\int_a^b cy\,dx = c\int_a^b y\,dx}$$

where c is a positive constant, is obtained by factoring out a c from each term $y\,dx$ of the sum on the left. As the following example shows, this equation is Cavalieri's principle expressed in Leibniz's notation.

Example 5.12. Consider the circle $x^2+y^2 = 5^2$. Solve for y to get $y = \pm\sqrt{25-x^2}$. Taking the positive value for y gives the equation $y = \sqrt{25-x^2}$ for the upper half of this circle. See Figure 5.18(a). It follows that $\int_{-5}^5 y\,dx = \int_{-5}^5 \sqrt{25-x^2}\,dx$ is equal to the area under the upper half of the circle. Since the radius is 5, we get $\int_{-5}^5 \sqrt{25-x^2}\,dx = \frac{1}{2}\pi 5^2 = \frac{25\pi}{2}$.

Now consider the ellipse $\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1$. Solving for y gives us $\frac{y^2}{2^2} = 1 - \frac{x^2}{5^2} = \frac{5^2-x^2}{5^2}$, or $y^2 = \frac{2^2}{5^2}(5^2-x^2)$. Take $y = \frac{2}{5}\sqrt{5^2-x^2}$ and note that the graph of this equation is the upper half of the ellipse. (See Figure 5.18(b).) So $\int_{-5}^5 \frac{2}{5}\sqrt{25-x^2}\,dx$ is the area under the upper half



(a) (b)

Figure 5.18

of the ellipse. By formula (*) above,

$$\int_{-5}^{5} \frac{2}{5}\sqrt{25 - x^2}\,dx = \frac{2}{5}\int_{-5}^{5}\sqrt{25 - x^2}\,dx$$
$$= \frac{2}{5} \cdot \frac{25\pi}{2} = 5\pi.$$

Therefore, the area of the full ellipse is $2(5\pi) = 10\pi$. This is a result that was achieved in Section 4.6 by use of Cavalieri's principle.

5.4 The Fundamental Theorem of Calculus

We turn next to a publication of Leibniz[4] from 1693. In it, Leibniz discusses a problem posed to him by a Parisian scholar:

> Claude Perrault ... who has ... distinguished himself with his studies in mechanics and architecture has presented to me the following problem. He had presented it to many others before me and openly admitted that he had not as yet been successful in solving it himself. A weight is placed on a horizontal plane. One end, say B, of a piece of string or small chain AB is attached to the weight. When the other end A of the string is moved along the fixed straight line AA' in the plane, so that the string AB lies in the plane for the duration of the motion, the weight is pulled along, and describes a curve BB'. The problem concerns the determination of this curve. [See Figure 5.19.] Perrault instead made use (as an illustration) of a pocketwatch with silver housing B, which he pulled across the table with the attached chain AB by leading the end A along a straight line AA'. In the process the lowest point of the housing (the midpoint of the bottom) described the curve BB'. I observed this curve with some care (I had been occupying myself primarily with the consideration of tangents) and immediately made the relevant remark that the string always touches the curve tangentially, e.g., the straight line $A''B''$ is the tangent of the curve BB' at B''.... I saw, therefore, that the question reduced to the following: To find a curve satisfying the condition that the section of the tangent AB between the axis AA' and the curve BB' is equal to a fixed constant. And it was not difficult for me to figure out that the description of this curve could be reduced to the quadrature of the hyperbola.

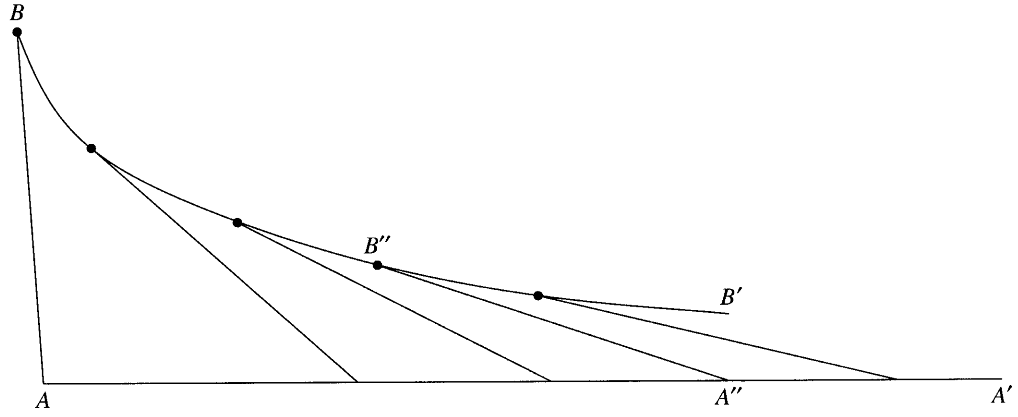


Figure 5.19

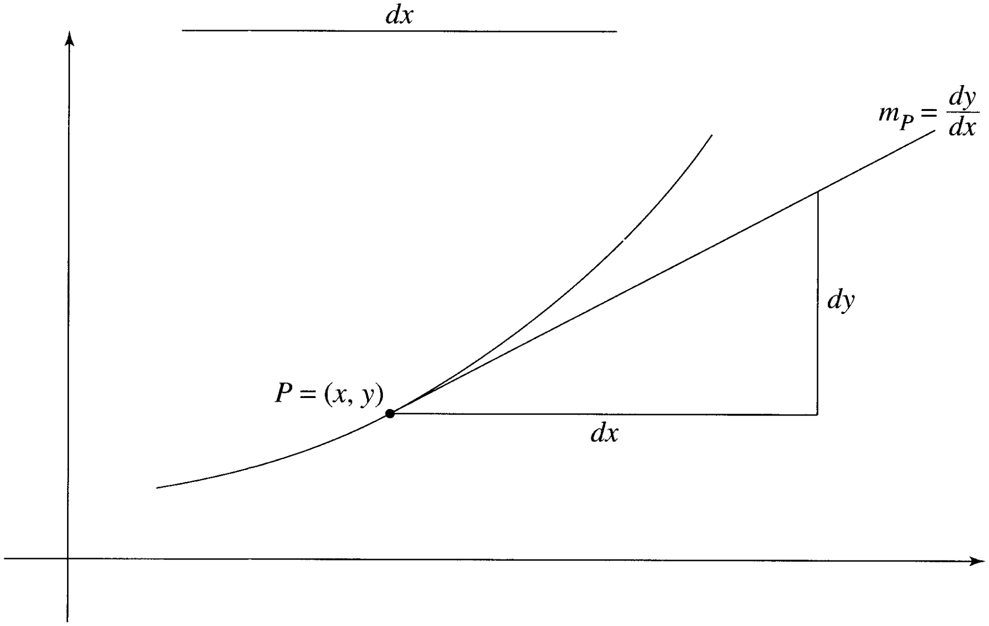


Figure 5.20

Since quadrature, or "squaring," refers to the determination of area. Leibniz has made reference to a fundamental relationship between and slopes of tangent lines and areas under curves. It is the purpose of the remainder of his article to bring this connection to light. We will describe the essence of what Leibniz does.

Let a curve be given and let $P = (x, y)$ be a point on the curve. Consider the tangent to the curve at P and let m_P be its slope; refer to Figure 5.20. Let dx be any length and form the triangle shown. As in the earlier discussion about areas, dx is often considered to be small. Observe that $\frac{dy}{dx} = m_P$, so that

$$dy = m_P \cdot dx.$$

Leibniz calls dx the *differential of* x and the related quantity $dy = m_P \cdot dx$ the *differential of* y. He calls the triangle formed by the segments of lengths dx and dy and the tangent the *characteristic triangle*. For example, in the case of the parabola $y = x^2$, the slope of the tangent is $m_P = 2x$, so that $dy = 2x \cdot dx$. In the case of the curve $y = x^3$ the slope of the tangent is $m_P = 3x^2$ and $dy = 3x^2 \cdot dx$. The construction pictured in Figure 5.20 will provide Leibniz with the link between areas under curves and slopes of tangent lines.

Leibniz considers a system of Cartesian coordinates and fixes a curve **C** above an interval from a to b on the x-axis. He now assumes that he has another curve **A** with the following property for all points x between a and b:

> The slope m_P of the tangent at the point P on **A** lying above x is equal to the y-coordinate of the point on **C** lying above x.

Figure 5.21 illustrates the connection between **C** and **A**: For any number x between a and b, let P and Q be the points on **A** and **C** respectively, both with first coordinate x. Then the slope m_P of the tangent to **A** at P is equal to the second coordinate of the point Q on **C**. So

$$m_P = y.$$

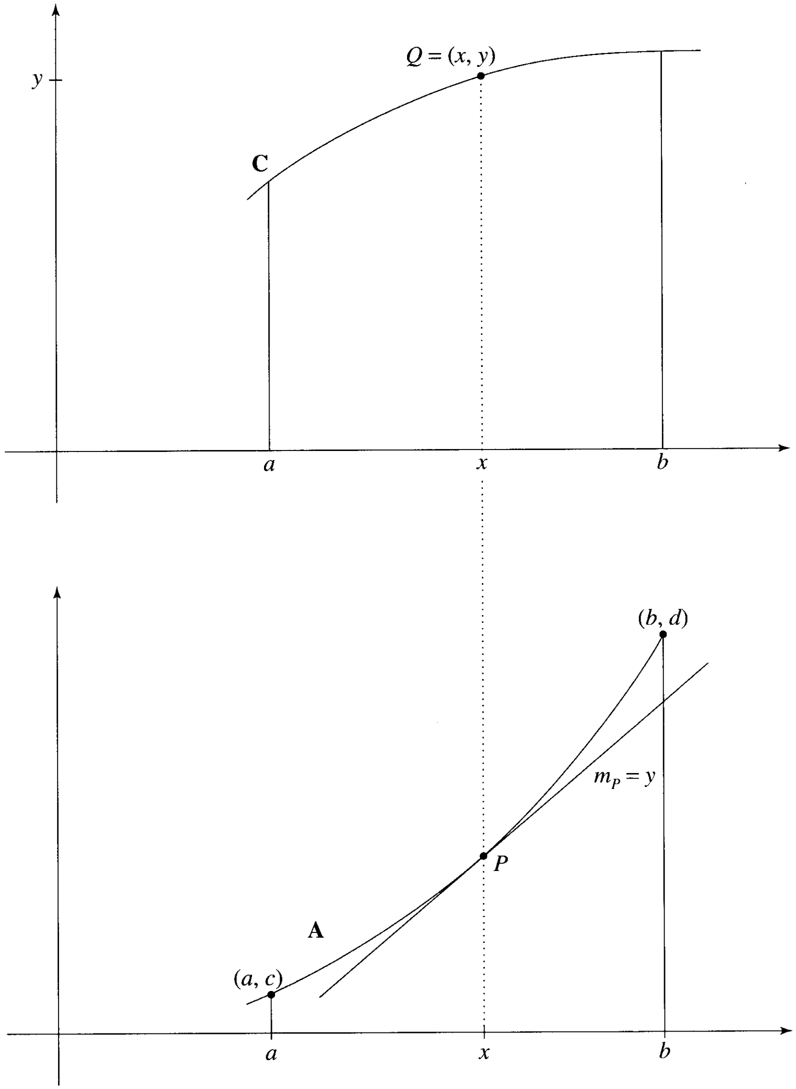


Figure 5.21

Next Leibniz takes any dx and places a characteristic triangle with sides dx and dy at the point P on **A**; see Figure 5.22. Observe that $\frac{dy}{dx} = m_P$ and therefore, that

$$\frac{dy}{dx} = y.$$

He then places a rectangle under the curve **C**, as shown (in black) in Figure 5.23. Since $y = \frac{dy}{dx}$, observe that the area $y \cdot dx$ of the rectangle is equal to the length dy of the vertical leg of the triangle at P. (How, you ask, is this equality possible, given that the rectangle is large and the segment tiny?)

Now pack lots of points between a and b. These determine, exactly as described in Section 5.3, many very thin rectangles under the curve **C**. As we just saw, the area $y \cdot dx$ of each rectangle is numerically

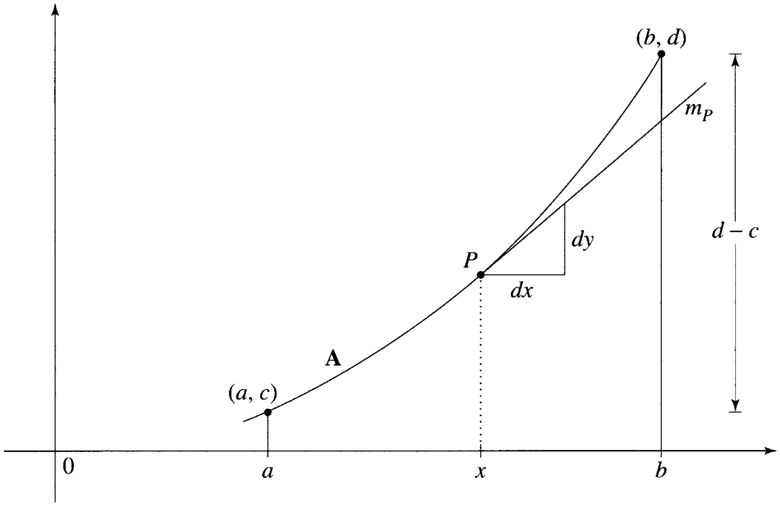


Figure 5.22

equal to the length of the corresponding segment dy. It is now plausible for Leibniz to conclude that the sum of the areas $y \cdot dx$ of all of the rectangles under **C** is equal to the sum of the lengths of all of the segments dy from c to d; see Figure 5.24. So Leibniz has shown that the area under **C** above the interval from a to b is equal to $d - c$. In his notation,

$$\int_a^b y\,dx = d - c.$$

This formula is known as the *Fundamental Theorem of Calculus.*

Let's consider a specific example to illustrate what Leibniz has accomplished.

Example 5.13. Consider the parabola $y = x^2$ over the interval $0 \le x \le 1$. This is the curve **C**. Recall from Section 5.2 that the slope of the tangent of $y = Ax^3$ at any point $P = (x, y)$ is equal to $m_P = 3Ax^2$. Taking $A = \frac{1}{3}$, we see that $m_P = x^2$. Therefore, the graph of $y = \frac{1}{3}x^3$ has the property that the slope of the tangent m_P at x is equal to the y-coordinate of the point on **C** at x. In other words, the graph of $y = \frac{1}{3}x^3$ satisfies what is required of the curve **A**. Since the point on **A** corresponding to $x = 0$ is $(0, 0)$ and that corresponding to $x = 1$ is $(1, \frac{1}{3})$, the roles of the points (a, c) and (b, d) are now played by $(0, 0)$ and $(1, \frac{1}{3})$. It follows that the area under $y = x^2$ over the interval $0 \le x \le 1$ is $d - c = \frac{1}{3} - 0 = \frac{1}{3}$. Notice that the discussion that follows Example 5.10 has already confirmed this.

5.5 Functions

Leibniz's analysis of the area under a curve implicitly required that for a given x there is at most one point on the curve above that x. Refer to Figure 5.14(b) and notice that if there were more than one such point, the entire matter would be ambiguous. In terms of the equation of the curve, the requirement is that for a given x, there is at most one y with the property that the pair (x, y) satisfies the equation. This condition is met by the equation $y = x^2$. For example, the only point (x, y) on this graph with $x = 2$ is $(2, 4)$, since $y = 2^2 = 4$. Similarly, when $x = 3$, y must be equal to $3^2 = 9$, and so on. A similar thing is true for the equations $y = x^3$, or $y = \sqrt{x}$, or $y = \frac{1}{x}$, or $y = \frac{1}{x^2}$. In each of these cases, we see that for a given x, there is at most one y that satisfies the equation. So a given x determines a single y. This property is the essen-

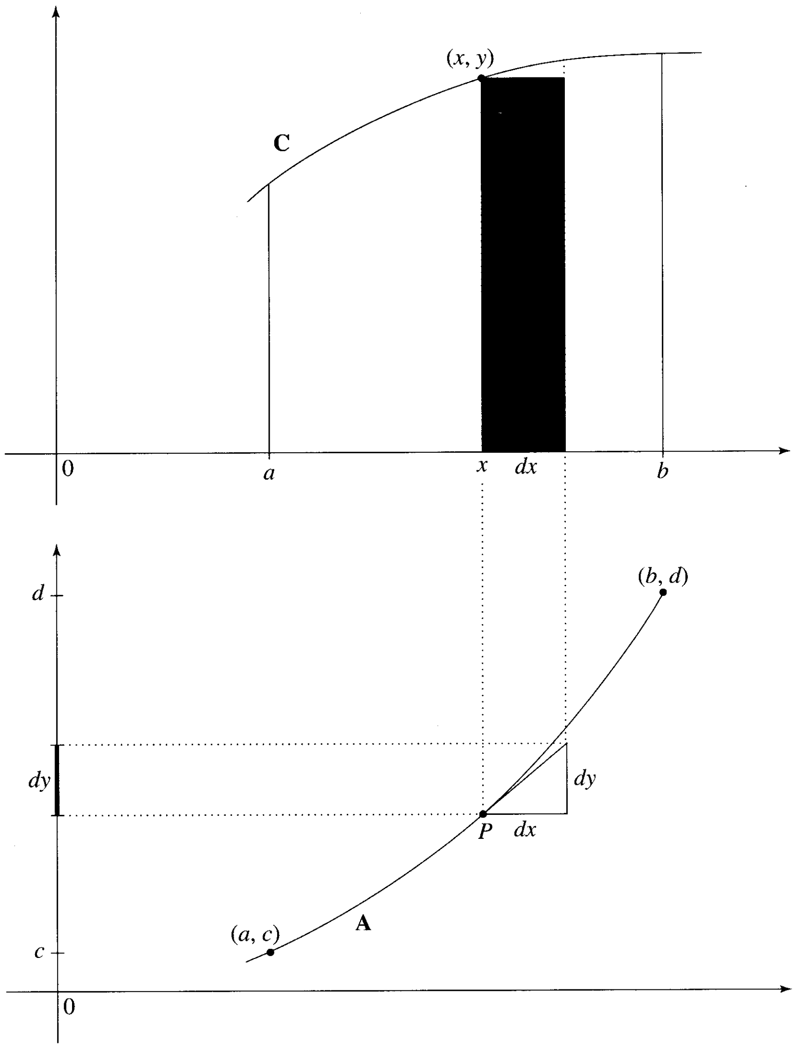


Figure 5.23

tial element in the concept of a function. Leibniz used the notion of a function only in the context of quantities related to curves. The more definitive concept presented next was not established until later.

A *function* is a rule that assigns exactly one real number to each number from a set of real numbers. Such a rule is often given by an algebraic expression. The rule $f(x) = 3x - 7$ is an example of a function. Note that $f(2) = 3 \cdot 2 - 7 = -1$ and $f(4) = 3 \cdot 4 - 7 = 5$. So f assigns -1 to the number 2, and 5 to the number 4, and $3x - 7$ to a typical real number x. Functions of the form $f(x) = 3x - 7$, $g(x) = -\frac{1}{2}x + 6$, or more generally, $h(x) = mx + b$, where m and b are constants, are called *linear functions*. Functions of the form $f(x) = 6$, or $g(x) = -7$, or $h(x) = b$, with b any constant, are called *constant functions*.

The *graph* of a function f is the set of all points (x, y) in the Cartesian plane that satisfy the equation

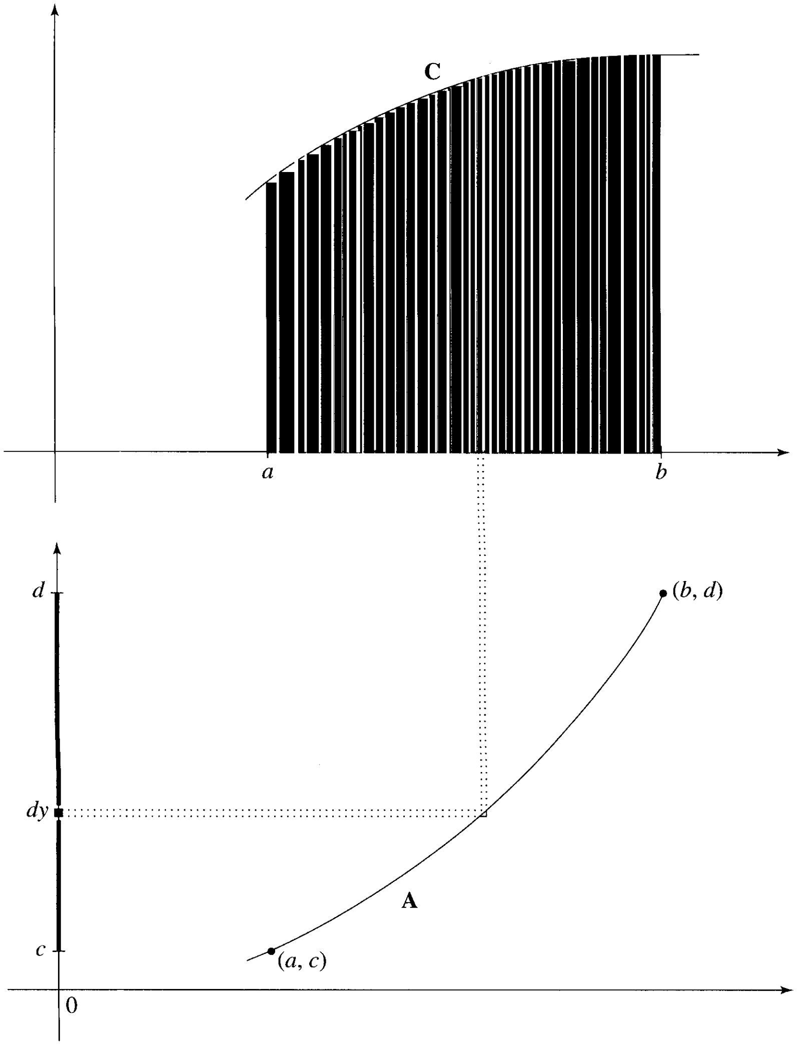


Figure 5.24

$y = f(x)$. So the graphs of the linear functions $f(x) = 3x - 7$, $g(x) = -\frac{1}{2}x + 6$, and $h(x) = mx + b$ are the graphs of the equations $y = 3x - 7$, $y = -\frac{1}{2}x + 6$, and $y = mx + b$, respectively. Observe that they are all lines.

Example 5.14. (a) The rule f that assigns the number x^2 to any x is the function $f(x) = x^2$. This rule is *defined*, i.e., makes sense, for all x. For example, $f(2) = 4$, $f(3) = 9$, and $f(-2) = 4$. So the points $(2,4)$, $(3,9)$, and $(-2,4)$ are all on the graph.

(b) The rule $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ is a function. Observe that $f(0) = 0, f(1) = 1, f(3) = \sqrt{3}$, and $f(4) = 2$. So the points $O = (0,0)$, $(1,1)$, $(3,\sqrt{3})$, and $(4,2)$ are on the graph. This function is defined only for $x \geq 0$. For example, $f(-2) = \sqrt{-2}$ is not defined, i.e., it does not make sense.

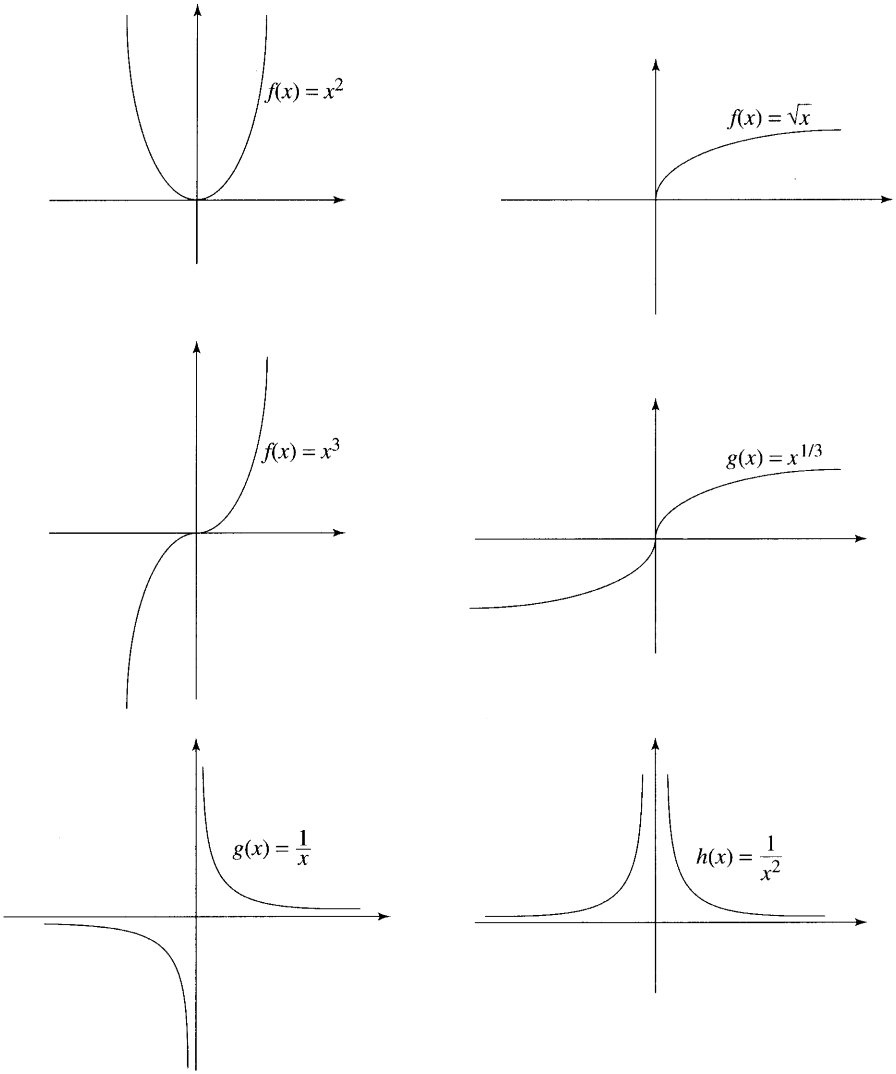


Figure 5.25

(c) The rule $f(x) = x^3$ is a function. It is defined for all x. Note that $(1,1)$, $(-1,-1)$, and $(-2,-8)$ are on the graph.

(d) The rule $g(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ is a function defined for all x. Check that $(-1,-1)$, $(8,2)$, and $(-27,-3)$ are on its graph.

(e) The rule g given by $g(x) = \frac{1}{x} = x^{-1}$ is a function. It is defined for all x except $x = 0$. The points $(1,1)$, $(2,\frac{1}{2})$, and $(\frac{1}{2},2)$ are on the graph.

(f) The rule $h(x) = \frac{1}{x^2} = x^{-2}$ is a function defined for all $x \neq 0$. Since $h(2) = \frac{1}{4}$, $h(-1) = 1$, and $h(3) = \frac{1}{3}$, the points $(2,\frac{1}{4})$, $(-1,1)$, and $(3,\frac{1}{3})$ are on the graph.

The graphs of the functions considered in (a)–(f) are sketched in Figure 5.25.

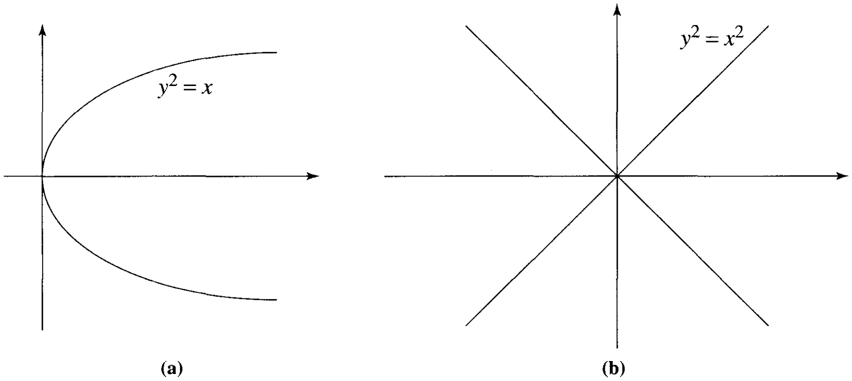


Figure 5.26

The functions $f(x) = x^2$, $g(x) = -x^2 + 3x - 4$, and, in general $h(x) = Ax^2 + Bx + C$, where $A \neq 0$, are called *quadratic functions*. Their graphs are the graphs of the equations $y = x^2$, $y = -x^2 + 3x - 4$, and $y = Ax^2 + Bx + C$, respectively. Note that the graph of any quadratic function is a parabola. (Refer to Section 4.3.)

Consider the equation $y^2 = x$. The pairs of numbers $x = 4, y = 2$ and $x = 4, y = -2$ are both solutions, so this equation does not determine y uniquely in terms of x. In general, for a given $x \geq 0$, both $y = \sqrt{x}$ and $y = -\sqrt{x}$ are solutions. Similarly, in reference to the equation $y^2 = x^2$, for a given x, both $y = x$ or $y = -x$ are solutions. Again, for a given x, there are two possibilities for y. Therefore neither equation defines a function of x.

A look at the graph of an equation in x and y makes this more explicit. If any vertical line crosses the graph at most once, then the equation defines a function of x. But if at least one vertical line crosses the graph twice or more, then it does not define a function of x. A comparison of the graphs of Figure 5.25 with those of Figure 5.26 illustrates the point.

The function concept clarifies the topics of this chapter. We will now incorporate it into our discussion and use it to reformulate some of the matters that were already developed. We begin with another look at the computation of the slope of a tangent line.

A. The Derivative. Let f be a function and fix a point $P = (x, y)$ on its graph. Since $y = f(x)$, it follows that $P = (x, f(x))$. Suppose the graph has a nonvertical tangent at P. Since P is the only point on the graph with this x-coordinate, we will designate the slope of the tangent by m_x instead of m_P as was done in Section 5.2. Now let $Q = (\bar{x}, \bar{y})$ be some other point on the graph; then $\bar{y} = f(\bar{x})$. Set $\bar{x} - x = \Delta x$ and $\bar{y} - y = \Delta y$. Note that $\bar{x} = x + \Delta x$ and $\bar{y} = f(\bar{x}) = f(x + \Delta x)$. So $Q = (x + \Delta x, f(x + \Delta x))$. See Figure 5.27. Observe that the slope of the line through P and Q is

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Since $\Delta y = \bar{y} - y = f(x + \Delta x) - f(x)$, this is the ratio $\frac{\Delta y}{\Delta x}$ that was studied in Section 5.2. When Δx is pushed to zero, the slope of this line closes in on the slope m_x of the tangent to the graph at $P = (x, f(x))$. In the notation of limits, this can be expressed as

$$m_x = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Now consider the rule that assigns the number m_x to a given number x. This rule defines a function

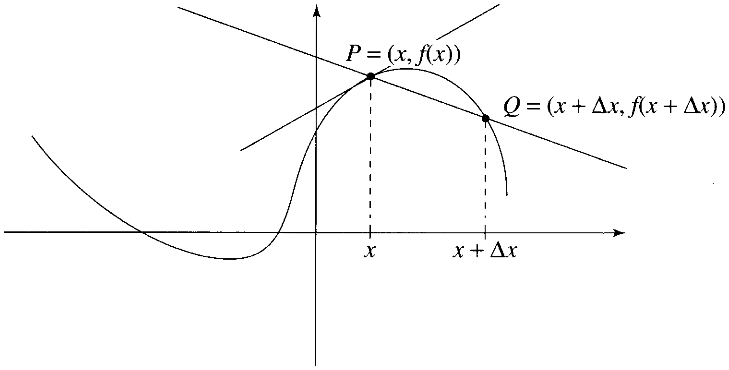


Figure 5.27

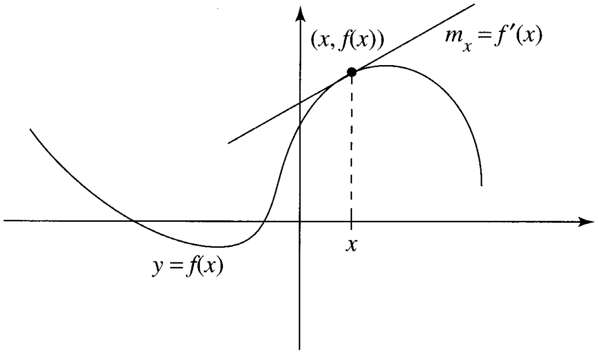


Figure 5.28

called the *derivative of f*, denoted by f'. To repeat, f' is the function given by the rule

$$f'(x) = m_x,$$

where m_x is the slope of the tangent to the graph of f at the point $(x, f(x))$. See Figure 5.28. Once more, the derivative of the function f is also a function. Its rule f' is derived from the rule of f in the manner just described.

Suppose that f is a linear function. Then it has the form $f(x) = mx + b$, with m and b constants. Its graph is a line with slope m. Take any point (x, y) on the graph. The tangent to the line is the line itself, so it follows that $m_x = m$. Therefore, $f'(x) = m$ for all x. Suppose that $m = 0$. Then $f(x) = b$, and $f'(x) = 0$ for all x. So the derivative of any constant function is the zero function.

Example 5.15. The derivative of $f(x) = \frac{1}{2}x + 1$ is $f'(x) = \frac{1}{2}$. The derivative of $f(x) = -3x + 6$ is $f'(x) = -3$. If $g(x) = 5$, then $g'(x) = 0$.

What are the derivatives of the functions of Examples 5.14(a)–14(f)?

Example 5.16. (a) Consider $f(x) = x^2$. Since

$$\begin{aligned} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \frac{(x + \Delta x)^2 - x^2}{\Delta x} \\ &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \frac{\Delta x(2x + \Delta x)}{\Delta x} \\ &= 2x + \Delta x, \end{aligned}$$

it follows that

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = 2x.$$

(b) Let $y = f(x) = \sqrt{x} = x^{\frac{1}{2}}$. We will not compute $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ directly (try it and you will probably encounter difficulties) but instead refer to Section 5.2. By squaring both sides, we get $y^2 = x$. The graph of $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ is the upper half of the graph of the parabola $x = y^2$; see Figure 5.26(a). It was shown in Section 5.2, using Leibniz's tangent method, that the slope of the tangent to this parabola at any point $P = (x,y)$ is $m_P = \frac{1}{2y}$. If the point P is on the graph of f, then $y = x^{\frac{1}{2}}$ and hence $m_x = m_P = \frac{1}{2}x^{-\frac{1}{2}}$. Therefore, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$.

(c) Let $f(x) = x^3$. It was shown in Section 5.2, using Leibniz's tangent method, that the slope of the tangent to the curve $y = x^3$ at any point $P = (x,y)$ is $m_P = 3x^2$. It follows that $f'(x) = 3x^2$. This derivative can also be established by proceeding as in (a). See Exercise 10.

(d) The derivative of $g(x) = x^{\frac{1}{3}}$ is $g'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. This is best shown by applying Leibniz's tangent method to the equation $y^3 = x$. See Exercise 8.

(e) The derivative of $g(x) = x^{-1}$ is $g'(x) = -\frac{1}{x^2} = -(x^{-2})$. This can be done as in (a) or with Leibniz's tangent method. See Exercise 7.

(f) The derivative of $h(x) = \frac{1}{x^2} = x^{-2}$ is $h'(x) = -2\frac{1}{x^3} = -2x^{-3}$. Do this by computing the limit $h'(x) = \lim\limits_{\Delta x \to 0} \dfrac{h(x+\Delta x) - h(x)}{\Delta x}$ directly. See Exercise 11.

Example 5.17. What is the slope of the tangent to the graph of $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ at the point $(2, \sqrt{2})$? This is computed by substituting $x = 2$ into $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$. So it is $f'(2) = \frac{1}{2}2^{-\frac{1}{2}} = \frac{1}{2\sqrt{2}}$. What about the slope of the tangent to the graph of $h(x) = \frac{1}{x^2}$ at $(-1, 1)$? This slope is computed by evaluating $h'(x) = -2x^{-3}$ at $x = -1$. Hence it is $h'(-1) = (-2)(-1) = 2$.

The notation $\frac{dy}{dx}$ of Section 5.4 is also used to denote the derivative of a function. So if $y = x^3$, then $\frac{dy}{dx} = 3x^2$. And if $y = \frac{1}{x^2}$, then $\frac{dy}{dx} = -2x^{-3}$.

Leibniz develops many of the basic properties of the derivative. We will turn to these in detail in Chapter 8. A few examples will do for now. Consider the function $f(x) = 5\sqrt{x} = 5x^{\frac{1}{2}}$. The derivative of f is equal to 5 times the derivative of the function $y = \sqrt{x}$. So by Example 5.16(b), $f'(x) = 5\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{5}{2\sqrt{x}}$. The point is that when computing the derivative of a function, constants can be factored out first. Another basic property asserts that the derivative of the sum (or difference) of two functions is equal to the sum (or difference) of their derivatives.

Example 5.18. Using Example 5.16, we see that the derivative of $y = \frac{1}{x^2} + x^3 + 4x$ is equal to $\frac{dy}{dx} = (-2)\frac{1}{x^3} + 3x^2 + 4$. In a similar way, if $f(x) = 2x^2 - \frac{5}{x} - 7$, then $f'(x) = 4x + 5\frac{1}{x^2}$.

B. Antiderivatives. Let f be a function. A function F that has the property that the derivative F' is equal to f is called an *antiderivative* of f.

Example 5.19. Let $F(x) = x^2$ and recall that $F'(x) = 2x$. Therefore $F(x)$ is an antiderivative of $f(x) = 2x$. Note that $G(x) = \frac{1}{3}x^3$ is an antiderivative of $g(x) = x^2$. Similarly, if $F(x) = x + 225\frac{1}{x}$, then $F'(x) = 1 - 225\frac{1}{x^2} = \frac{x^2-225}{x^2}$. Therefore, $F(x) = x + 225\frac{1}{x}$ is an antiderivative of $f(x) = \frac{x^2-225}{x^2}$.

Let's consider the situation of a function h that has the property that $h'(x) = 0$ for all x. This means that the slope of the tangent at any point P on the graph is zero. So all tangents are horizontal. If the graph were to curve upward or downward at any point P, the tangent line would either rise or fall; see Figure 5.29. Since this does not happen, the only possibility is that the graph is a horizontal line. Because every horizontal line has the form $y = C$ for some constant C, it follows that $h(x) = C$ for all x. In words, a function whose derivative is zero must be a constant function.

There is an important consequence of this. Suppose that $y = F(x)$ and $y = G(x)$ are both antiderivatives of the same function $f(x)$. Since

$$F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x),$$

it follows that the derivative of $G(x) - F(x)$ is equal to

$$G'(x) - F'(x) = 0.$$

Therefore, by the fact just pointed out, $G(x) - F(x)$ is equal to a constant C; so

$$G(x) = F(x) + C.$$

Suppose that one antiderivative of a function f is known. We have shown that any other antiderivative

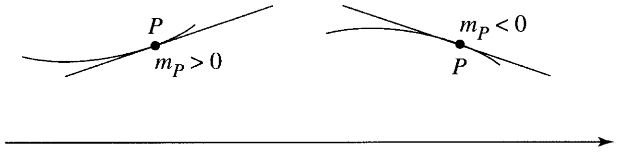


Figure 5.29

of f can obtained by adding a constant to the known antiderivative.

Let's return to the Fundamental Theorem of Calculus of Section 5.4. Let f be a function. Restrict x to lie in an interval $a \leq x \leq b$ and suppose that $y = f(x) \geq 0$ for all such x. As in Section 5.3, pack lots of points into the interval from a to b and form the products $y \cdot dx = f(x) \cdot dx$. Recall that

$$\int_a^b y\, dx = \int_a^b f(x)\, dx$$

is the sum of all these products.

Suppose that the function F is an antiderivative of f. So $F'(x) = f(x)$ for all x. Let **A** be the graph of F and let $P = (x, y)$ be any point on **A**. Since $F'(x) = m_P$ is the slope of the tangent to **A** at the point P, notice that

$$F'(x) = m_P = f(x).$$

Therefore, the graph **A** of F fulfills the requirements of Leibniz's development of the Fundamental Theorem of Calculus. Refer back to Figure 5.21. Because (a, c) and (b, d) are on the graph **A** of F, it follows that $F(a) = c$ and $F(b) = d$. The equality $\int_a^b y\, dx = d - c$ of Section 5.4 can therefore be rewritten

$$\boxed{\int_a^b f(x)\, dx = F(b) - F(a)}$$

In summary, this equality means this: To evaluate a sum of the form $\int_a^b f(x)\, dx$, find an antiderivative F of f and compute the difference $F(b) - F(a)$. The expression $\int_a^b f(x)\, dx$ is called the *definite integral* of the function f from a to b. It is now customary to use the notation $F(x)\big|_a^b$ for the difference $F(b) - F(a)$.

5.6 Some Applications

We continue with a glimpse at some applications of the derivative. Consider a function f and its graph. Have a look at Figure 5.30 and consider a high point on the graph (the "top of a peak"). If c_1 is the x-coordinate of such a point, then the y-coordinate $f(c_1)$ is larger than $f(x)$ for any x near c_1. Observe that the slope $f'(c_1)$ of the tangent is zero. In the same way, if c_2 is the x-coordinate of a low point on the graph (the "bottom of a valley"), then the y-coordinate $f(c_2)$ is smaller than $f(x)$ for any x near c_2. Here too, $f'(c_2)$ is zero. This suggests that the solution of a problem that calls for the determination of a maximum or minimum value of a function should focus on the values of x at which the derivative is equal to zero. We will illustrate this by considering two problems of the type considered early in the 17th century.

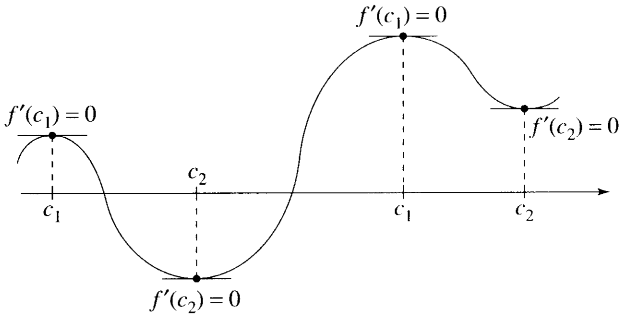


Figure 5.30

A. Finding Maximum and Minimum Values. Consider a segment that is cut into two pieces as shown in Figure 5.31.

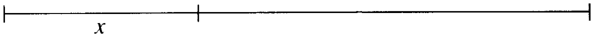


Figure 5.31

Example 5.20. Suppose that the total length of the segment is 1200 units. How is the segment to be cut so that the product of the lengths of the two pieces is as large as possible?

Solution. If x is the length of one of the pieces, then the other has length $1200 - x$. The product of the lengths is given by the function $f(x) = x(1200 - x) = 1200x - x^2$. The question is: What is the largest value of this function? Since $f'(x) = 1200 - 2x$, we see that $f'(x) = 0$ when $x = 600$. Because the graph of f is a parabola that opens downward, it follows that the product $f(x) = x(1200 - x)$ is largest when $x = 600$. Since $1200 - 600 = 600$, the length of the other piece is also 600 units.

Example 5.21. Suppose that the product of the lengths of the two pieces in Example 5.20 is specified to be 225. What is the shortest segment for which this is possible?

Solution. Let y be the length of the entire segment. Given that one piece has length x, the second piece has length $y - x$. The product of the lengths is $(y - x)x = 225$. Therefore, $yx - x^2 = 225$. So $yx = x^2 + 225$, and hence $y = x + \frac{225}{x}$. We need to find the minimum value of $f(x) = x + 225\frac{1}{x}$. By Example 5.16(e) and the properties of derivatives, $f'(x) = 1 - 225\frac{1}{x^2} = \frac{x^2-225}{x^2}$. In order for this to be equal to zero, the numerator $x^2 - 225$ must be zero. This happens when $x = \pm 15$. Since x is length, $x = 15$, and $f(15) = 15 + \frac{225}{15} = 15 + 15 = 30$. Experiment with other lengths and observe that 30 is the smallest possible length of the segment.

We conclude our discussion of the calculus of Leibniz by showing that the summation process that gives rise to the definite integral can be applied to problems other than the computation of area. We will illustrate this by computing volumes and lengths of curves.

B. Volumes. Begin with the fact that the volume of a cylinder is equal to the area of the base times the height. So if the height is h and the circular base has radius r, then the volume is $\pi r^2 h$; see Figure 5.32.

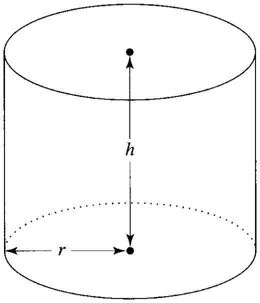


Figure 5.32

Now consider a semicircle of radius r and place a coordinate system as shown in Figure 5.33. As we have already done a number of times, pump the interval from 0 to $2r$ full of points, and consider the corresponding thin rectangles under the circle. A typical one is shown. It is placed at x, it is y units high, and it has thickness dx. Rotate the semicircular region one complete revolution about the x-axis and note that it sweeps out a sphere S of radius r. The thin rectangle sweeps out a thin cylindrical disc of radius y and thickness (height) dx. The circular base of this thin cylinder has area πy^2 and its volume is $\pi y^2 dx$. Since the point (x, y) satisfies $(x-r)^2 + y^2 = r^2$, observe that $y^2 = r^2 - (x-r)^2 = 2rx - x^2$. So the volume of the thin cylinder is

$$\pi y^2\, dx = (2\pi rx - \pi x^2)\, dx.$$

Do this for all the rectangles from 0 to $2r$. Since all of the thin rectangles together fill out the semicircular region, it follows that the sum of the volumes of all the corresponding thin cylinders is the volume of the entire *solid of revolution*, i.e., the volume of the sphere S of radius r. It is now important to observe that this

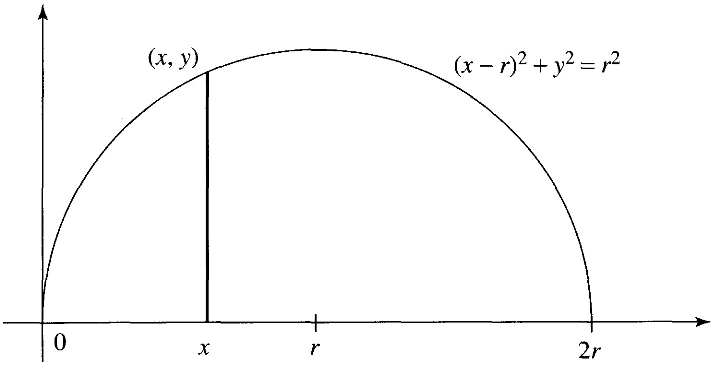


Figure 5.33

sum, namely, the sum of all the terms $(2\pi rx - \pi x^2)\,dx$, is precisely a sum of the type considered in Section 5.3. Therefore,

$$\text{Volume of } S = \int_0^{2r} (2\pi rx - \pi x^2)\,dx$$
$$= \pi \int_0^{2r} (2rx - x^2)\,dx.$$

Refer to the end of the previous section for the fact that this sum can be evaluated as follows: If $y = G(x)$ is an antiderivative of the function $g(x) = 2rx - x^2$, then

$$\pi \int_0^{2r} (2rx - x^2)\,dx = \pi \left[G(x)\Big|_0^{2r}\right] = \pi[G(2r) - G(0)].$$

Because the function rx^2 is an antiderivative of $2rx$, and $\frac{1}{3}x^3$ is an antiderivative of x^2, it follows that $G(x) = rx^2 - \frac{1}{3}x^3$ is an antiderivative of $g(x) = 2rx - x^2$. Since

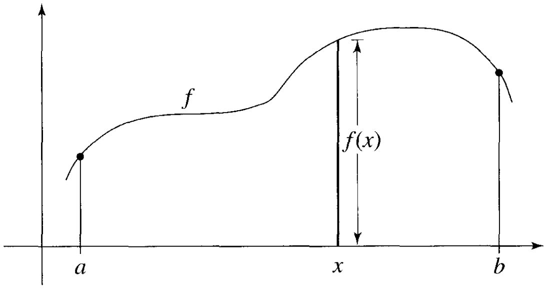


Figure 5.34

$G(2r) - G(0) = r(2r)^2 - \frac{1}{3}(2r)^3$,

$$\text{Volume of } S = \pi \int_0^{2r} (2rx - x^2)\,dx$$
$$= \pi \left(r(2r)^2 - \frac{1}{3}(2r)^3 \right) = 4\pi r^3 - \frac{8}{3}\pi r^3$$
$$= \frac{4}{3}\pi r^3.$$

Leibniz's summation method combined with the Fundamental Theorem of Calculus has shown the volume of a sphere S of radius r to be $\frac{4}{3}\pi r^3$.

Using this basic argument, many volumes of revolution can be computed. Let f be a function that satisfies $f(x) \geq 0$ for all x between a and b on the x-axis. A typical situation is sketched in Figure 5.34. Proceed exactly as in the situation of the semicircle. The interval from a to b is packed with points, and these determine thin rectangles under the curve. A typical one is shown. It is placed at x, it has height $f(x)$ and thickness dx. Now rotate the region bounded by the graph, the x-axis, and the lines $x = a$ and $x = b$ for one complete revolution about the x-axis. The volume of the resulting solid is obtained by adding the volumes of all of the thin cylindrical discs determined by the rotation of the rectangles. The typical disc shown has circular area $\pi \cdot (f(x))^2$ and height dx, so it has volume $\pi \cdot (f(x))^2 \cdot dx$. Since the volume V of the solid is the sum of all of the volumes of these discs, it is given by

$$\boxed{V = \int_a^b \pi(f(x))^2\,dx}$$

Notice that this definite integral is also equal to the area under the graph of the function $\pi(f(x))^2$ from a to b.

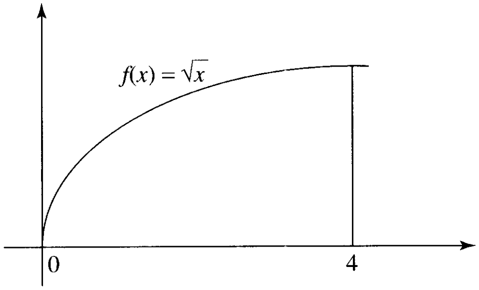


Figure 5.35

Example 5.22. Revolve the region shown in Figure 5.35 one complete revolution about the x-axis and compute the volume V of the solid thus obtained. Substituting $f(x) = \sqrt{x}$ into the formula just developed, we find that

$$V = \int_a^b \pi(f(x))^2\,dx = \int_0^4 \pi(\sqrt{x})^2\,dx = \pi \int_0^4 x\,dx.$$

Since $G(x) = \frac{1}{2}x^2$ is an antiderivative of $g(x) = x$, it follows that $V = G(x)\big|_0^4 = G(4) - G(0) = \frac{1}{2} \cdot 4^2 = 8$.

C. Lengths of Curves. The definite integral can also be applied to the computation of the lengths of curves. This is done as follows. Let f be a function, and let $P = (a, c)$ and $Q = (b, d)$ be two points on its graph; see Figure 5.36. Here is the strategy for computing the length L of the curve between the points P and Q.

Again pump the interval from a to b full of points, packing them tightly together. Take a typical point x and let $x + dx$ be the very next one. Let (x, y) be the point on the graph above x, and use a segment of length dx to construct a characteristic triangle at (x, y). Figure 5.37 shows what is happening under a microscope. Notice that the slope of the tangent at (x, y) is $\frac{dy}{dx}$. In particular, $f'(x) = \frac{dy}{dx}$. By the Pythagorean theorem, the length of the hypotenuse of the characteristic triangle is $\sqrt{(dx)^2 + (dy)^2}$. Factoring out $(dx)^2$, we get

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2}$$

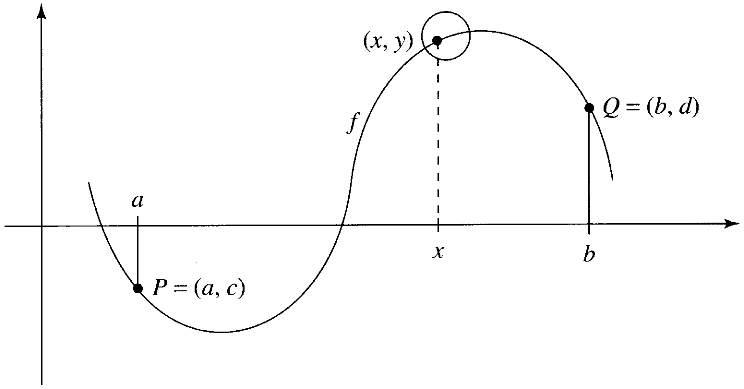


Figure 5.36

$$= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+(f'(x))^2}\,dx.$$

Since dx is extremely small, the hypotenuse of the triangle is for all computational purposes equal to the length of the portion of the graph of $f(x)$ inside the triangle. The length L of the curve from P to Q is the sum of the lengths of all of these arcs as x ranges from a to b. Thus, the length L is given as the sum of all of the terms $\sqrt{1+(f'(x))^2}\,dx$ as x ranges from a to b. Since this sum is of the type considered in Section 5.3, we conclude that

$$\boxed{L = \int_a^b \sqrt{1+(f'(x))^2}\,dx}$$

By the Fundamental Theorem of Calculus, L can now be determined by finding an antiderivative G of the function $g(x) = \sqrt{1+(f'(x))^2}$ and evaluating $L = G(b) - G(a)$.

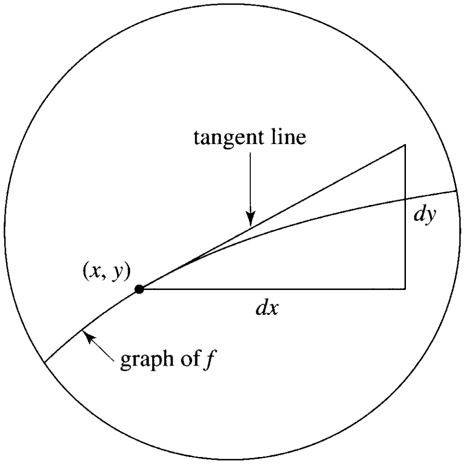


Figure 5.37

Let's apply our formula to the simplest case: the length of a line segment.

Example 5.23. Consider the function $f(x) = 2x - 3$ with the points $P = (-1,-5)$ and $Q = (5,7)$ on its graph, as pictured in Figure 5.38. We already know that the distance between P and Q is

$$\sqrt{(-1-5)^2+(-5-7)^2} = \sqrt{36+144} = \sqrt{180} = 6\sqrt{5}.$$

Let's see whether definite integrals provide the same result. Since $f'(x) = 2$, $a = -1$, and $b = 5$,

$$\int_a^b \sqrt{1+(f'(x))^2}\,dx = \int_{-1}^5 \sqrt{1+2^2}\,dx = \int_{-1}^5 \sqrt{5}\,dx.$$

Because $G(x) = \sqrt{5}x$ is an antiderivative of $\sqrt{5}$,

$$\int_{-1}^5 \sqrt{5}\,dx = G(5) - G(-1) = 5\sqrt{5} - (-1)\sqrt{5} = 6\sqrt{5},$$

as expected.

Example 5.24. Consider the circle $x^2 + y^2 = 5^2$ of radius 5 and with center the origin. Since $y = \pm\sqrt{5^2 - x^2}$, any point (x, y) on the upper half of the circle satisfies $y = \sqrt{5^2 - x^2}$. It follows that the graph of the function $f(x) = \sqrt{5^2 - x^2}$ is the upper half of the circle. See Figure 5.39. Consider the points P and Q on the circle with x-coordinates $-\frac{5}{2}\sqrt{2}$ and $\frac{5}{2}\sqrt{2}$, respectively. What is the length L of the circular arc from P to Q? By the formula already developed,

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx = \int_{-\frac{5}{2}\sqrt{2}}^{\frac{5}{2}\sqrt{2}} \sqrt{1+(f'(x))^2}\,dx.$$

Recall from the discussion preceding Example 5.8 of Section 5.2 that the slope of the tangent line to this circle at any point (x, y) is $-\frac{x}{y}$. It follows that $f'(x) = -\frac{x}{y} = -\frac{x}{\sqrt{5^2-x^2}}$. So $(f'(x))^2 = \frac{x^2}{5^2-x^2}$, and hence $1 + (f'(x))^2 = \frac{5^2-x^2+x^2}{5^2-x^2} = \frac{5^2}{5^2-x^2}$. Therefore,

$$L = \int_{-\frac{5}{2}\sqrt{2}}^{\frac{5}{2}\sqrt{2}} \sqrt{\frac{5^2}{5^2-x^2}}\,dx = \int_{-\frac{5}{2}\sqrt{2}}^{\frac{5}{2}\sqrt{2}} \frac{5}{\sqrt{5^2-x^2}}\,dx.$$

This definite integral is not easy to evaluate. We can, however, compute the length of arc PQ directly. Look again at Figure 5.39 and consider the angle determined by the segment OQ and the positive x-axis. The cosine of this angle is $\frac{\frac{5}{2}\sqrt{2}}{5} = \frac{\sqrt{2}}{2}$, so the angle is $\frac{\pi}{4}$. By symmetry, the angle determined by the segment OP and the negative x-axis is also $\frac{\pi}{4}$. It follows that $\angle POQ = \frac{\pi}{2}$. It is now clear that arc PQ is precisely one quarter of the circle of radius 5; so its length is $\frac{1}{4} \cdot 2\pi \cdot 5 = \frac{5}{2}\pi$. We have verified that

$$L = \int_{-\frac{5}{2}\sqrt{2}}^{\frac{5}{2}\sqrt{2}} \frac{5}{\sqrt{5^2 - x^2}}\, dx = \frac{5}{2}\pi.$$

Notice that this definite integral is also equal to the area under the graph of the function $\frac{5}{\sqrt{5^2-x^2}}$ from $-\frac{5}{2}\sqrt{2}$ to $\frac{5}{2}\sqrt{2}$.

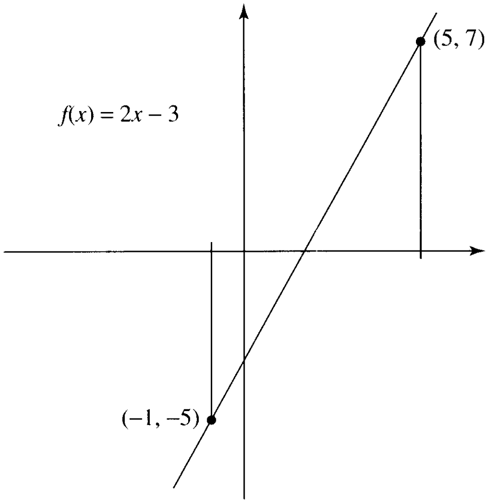


Figure 5.38

We have discussed three applications of definite integrals, namely, the computation of areas, volumes of revolution, and lengths of arcs. At least in principle, these can all be solved by using the Fundamental Theorem of Calculus. A variety of problems of mathematics and physics can be solved in this way. A word of caution is in order, however; the required antiderivatives are not always easily determined.

5.7 Postscript

The difference between the approaches of Leibniz (and also Newton as we will see in the next chapter) and those of Archimedes should now be clear: Archimedes developed ingenious methods (many of which contain the elements of what is now called calculus) to solve specific problems. His computation of the area of a parabolic section is a case in point. Leibniz and Newton, on the other hand, put together a general theory that is applicable to a wide range of problems. It must be pointed out that their achievements incorporated the work of many of their predecessors, among them Descartes, Cavalieri, and Kepler.

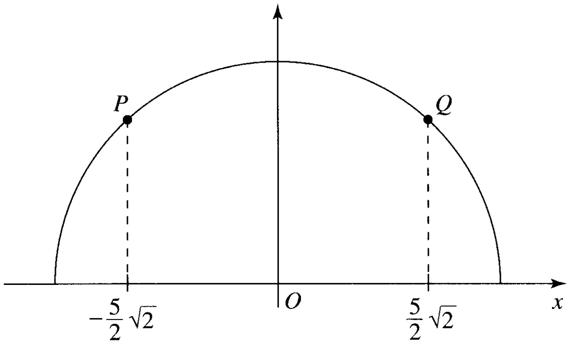


Figure 5.39

The description of Leibniz's work given in this chapter has retained the essence of his arguments, but it incorporates clarifications and improvements that came later. Functions, as already indicated, are implicit rather than explicit in Leibniz's work. They began to move to a central position in mathematics in the 1740s with Leonhard Euler. For Euler, a function was an "analytical expression composed in any way" from x and from "numbers or constant quantities." Later, in the work of Lagrange, the slope, in the role of the derivative, was treated as a function, and the notation $f'(x)$ was first used. The notion of the limit, which D'Alembert emphasized in the 1750s, and the notation $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, first used by Cauchy in the 1820s, both added clarity.

Leibniz's fame as a philosopher and scientist spread throughout Europe. He was in correspondence with most of the important European scholars of the day. He was named to the French Academy of Sciences, and the Habsburg emperor bestowed on him the title of baron. But Leibniz's last years were unhappy ones. His long dispute over the priority of the creation of the calculus with Newton and his English colleagues became increasingly bitter. When the son of his patron became King George I of England in 1714 and moved his court to London, Leibniz remained behind in Hanover in isolation. His work was neither understood nor appreciated. He suffered increasingly from illness and died in 1716.

Exercises

5A. Lines and Their Equations

1. What is the slope of the line determined by the points $(2,-3)$ and $(-6,2)$. Write an equation for this line.

2. Write an equation for the line with slope -3 and y-intercept 4.

3. Write an equation for the line that has slope $\frac{1}{2}$ and has the point $(3,-2)$ on its graph.

4. What are the slope and y-intercept of the line that has equation $2x + 7y + 2 = 0$?

5B. Computing Slopes of Tangents

Go through Leibniz's tangent method ("push Δx to zero") to compute the slope m_P of the tangent to the given curve at the given point P.

5. The parabola $y = x^2$ at the point $P = (2,4)$.

6. The curve $y = x^3$ at the point $P = (2,8)$.

7. The hyperbola $y = \frac{1}{x}$ at any point $P = (x,y)$.

8. The curve $x = y^3$ at any point $P = (x,y)$. [Hint: What happens to $\frac{\Delta y}{\Delta x}\Delta y$ when Δx is pushed to zero? Refer to Figure 5.11 in Section 5.2 and the surrounding discussion.]

9. The ellipse $\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$ at $P = (x,y)$. [If you get $m_P = -\frac{4^2}{5^2} \cdot \frac{x}{y}$, you are right.]

5C. Derivatives

10. Let $f(x) = x^3$. Determine $f'(x)$ by evaluating

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

11. Consider $h(x) = \frac{1}{x^2}$. Show that $h'(x) = \frac{-2}{x^3}$ by evaluating $\lim\limits_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x}$. [Hint: Take common denominators and cancel.]

12. Use the derivative formulas already developed to compute the slopes of the tangents to the graphs of the following functions at the given points.

- **i.** $f(x) = x^3$ at $(-2,-8)$.
- **ii.** $g(x) = \sqrt[3]{x}$ at $(-3, \sqrt[3]{-3})$.
- **iii.** $f(x) = \frac{1}{x}$ at $(-\frac{1}{3}, -3)$.
- **iv.** $f(x) = \frac{1}{x^2}$ at $(-2, \frac{1}{4})$.

13. By using facts from the text, compute the derivatives of each of the following functions:

- **i.** $f(x) = -10$.
- **ii.** $y = 4x + 7$.
- **iii.** $f(x) = 7x^2 - 5x + 2$.
- **iv.** $y = 2\sqrt[3]{x} + \pi x^3$.
- **v.** $g(x) = \frac{3}{x} + 3x - 6$.
- **vi.** $f(x) = 2x^3 + 3x + 4 - \frac{1}{x^2}$.

14. Consider the parabolic section determined by the parabola $y = -x^2 + 8x$ and the x-axis.

- **i.** Determine the point on the parabola with the property that the slope of the tangent line is zero.
- **ii.** Use Archimedes's theorem (not calculus) to compute the area of the parabolic section.

5D. Definite Integrals

15. Consider the graph of the equation $y = \frac{1}{x}$ over the interval from 2 to 4. Insert the points

$$2 \le 2.3 \le 2.5 \le 2.9 \le 3.4 \le 3.6 \le 4$$

and compute the sum of all of the rectangles $y \cdot dx$.

16. Consider the graph of the equation $y = \sqrt{x}$ over the interval from 0 to 2. Insert the points

$$0 \le \frac{1}{9} \le \frac{2}{9} \le \frac{4}{9} \le \frac{7}{9} \le \frac{11}{9} \le \frac{16}{9} \le 2$$

and compute the sum of all of the rectangles $y \cdot dx$. What is the actual value of the area under the graph of $y = \sqrt{x}$ from $x = 0$ to $x = 2$?

17. Explain with a diagram the full meaning of $\int_0^3 x^2\,dx$ and $\int_3^{12} \sqrt{x}\,dx$.

18. Draw a graph of the hyperbola $\frac{x^2}{5^2} - \frac{y^2}{4^2} = 1$. Then graph the equation $y = \frac{4}{5}\sqrt{(x^2 - 5^2)}$ and explain the full meaning of $\int_7^{10} \frac{4}{5}\sqrt{x^2 - 5^2}\,dx$.

5E. The Tractrix

19. Recall Perrault's problem from the beginning of Section 5.4. Tie a weight B to the end of a string and place it and the weight on a horizontal Cartesian plane. Stretch the string along the positive x-axis so that one end is at the origin O and the other end, where B is tied, is at $(a, 0)$. So a is the length of the string. Now move the end of the string (the one at the origin) up the positive y-axis. The curve that B traces out in the process is called a *tractrix*. Let (x, y) represent a typical position of the weight B. See Figure 5.40. Leibniz observed that the string is tangent to the curve at (x, y). Study the figure and express the slope m_x of the tangent to the curve at the point (x, y) as a function of x. Suppose that the tractrix is the graph of a function $y = f(x)$. What equation must the derivative of this function satisfy?

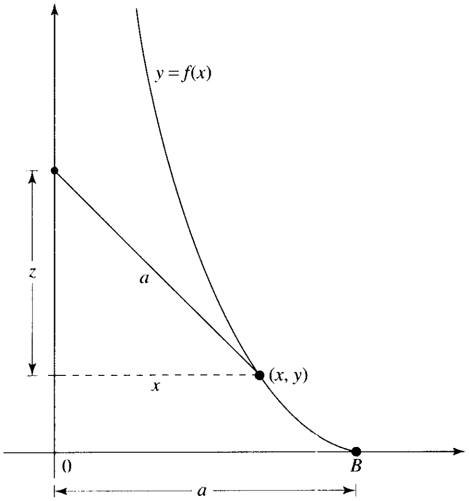


Figure 5.40

20. Use the results of Exercise 19 to express as a definite integral the length of the tractrix from the point $(a, 0)$ to some other point (c, d).

5F. Maximum and Minimum Values

In Exercises 21 and 22 make use of Figure 5.31.

21. If the segment is 1200 units long and the product of the squares of the lengths of the two pieces is to be as large as possible, how long should the two pieces be?

22. The product of the lengths of the two pieces is specified to be 300. What is the shortest segment with which this can be done?

23. Find the dimensions of the rectangle of largest area whose perimeter is 1000 cm.

24. Find that point on the parabola $y = x^2 + 1$ that is closest to the point $(3, 1)$. [Hints: Find that point on the parabola such that the square of its distance to $(3, 1)$ is minimal. Also, recall that if a is a root of a polynomial, then $x - a$ divides it.]

25. Consider the triangular region determined by the graph of the line $y = 3 - x$. Inscribe a rectangle as shown in Figure 5.41. When both the triangle and the rectangle are revolved one complete revolution about the x-axis, a cone and an inscribed cylinder result. Determine the largest volume this cylinder can have.

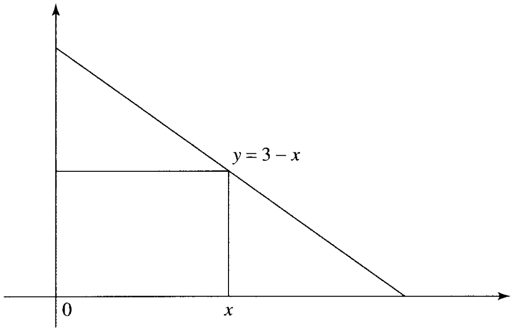


Figure 5.41

5G. Areas and Definite Integrals

26. Use the Fundamental Theorem of Calculus to evaluate the definite integrals

$$\int_0^3 x^2\,dx, \quad \int_{-8}^{-2} \frac{1}{x^2}\,dx, \quad \text{and} \quad \int_3^{12} \sqrt{x}\,dx.$$

Sketch the areas that you have determined.

27. The equation of the circle with radius 2 and centered at the origin is $x^2 + y^2 = 4$. Why is $y = \sqrt{4 - x^2}$ an equation whose graph is the upper semicircle only? Evaluate the definite integral

$$\int_0^2 \sqrt{4 - x^2}\,dx$$

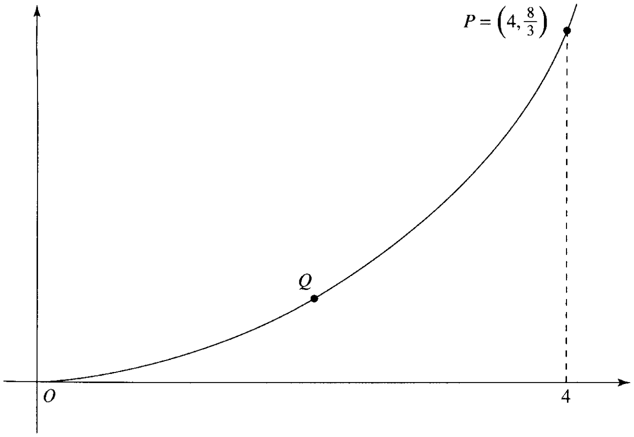


Figure 5.42

after drawing a picture of the area that it represents. Let a be a positive number and evaluate $\int_0^a \sqrt{a^2 - x^2}\,dx$ in the same way.

28. Evaluate the definite integral $\int_0^5 \frac{5}{2}\sqrt{5^2 - x^2}\,dx$ by making use of the area of an appropriate circle.

29. Let a and b be positive numbers. Use Exercise 27 and a basic property of the definite integral to evaluate $\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$. What area does this definite integral represent?

30. The graph in Figure 5.42 is that of the parabola $y = \frac{1}{6}x^2$ over the interval from 0 to 4.

- **i.** Determine the slope of the segment from O to $P = (4, \frac{8}{3})$. Determine the coordinates of the point Q with the property that the tangent to the curve at Q is parallel to OP.
- **ii.** Determine first the slope and then the equation of the line perpendicular to OP that goes through the point Q.
- **iii.** Consider the triangle OQP. Taking OP as the base, determine its height. What is the area of the triangle OQP?
- **iv.** Use Archimedes's theorem for the area of a parabolic segment to determine the area under the parabola from $x = 0$ to $x = 4$.
- **v.** Verify this answer by using the Fundamental Theorem of Calculus.

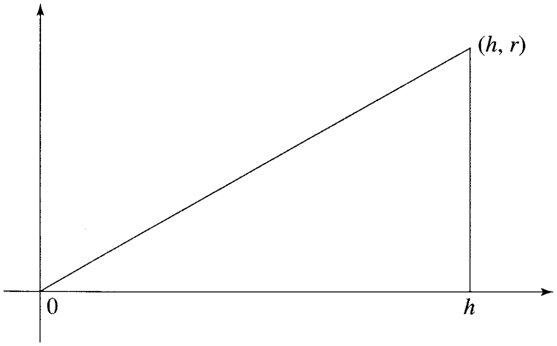


Figure 5.43

5H. Definite Integrals as Areas, Volumes, and Lengths of Curves

31. Use the Fundamental Theorem of Calculus to compute the volume of a cone of height h and base of radius r. [Hint: Rotate the triangular region in Figure 5.43.]

32. Determine the volume of the solid obtained by rotating the region above the x-axis and under the graph of the function $y = \sqrt{x}$, $0 \le x \le 3$, one complete revolution about the x-axis.

33. Show that the length of the parabolic arc on the parabola $y = x^2$ from the point $(2,4)$ to the point $(5,25)$ is equal to the area under the upper half of the hyperbola $y^2 - 4x^2 = 1$ from $x = 2$ to $x = 5$.

34. Consider the upper half of the ellipse $\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$ in Figure 5.44. Express as definite integrals:

- **i.** The area of the upper half of the ellipse.
- **ii.** The volume of the solid obtained by rotating it one revolution about the x-axis.
- **iii.** The length of the arc from $(1, \frac{8}{5}\sqrt{6})$ to $(3, \frac{16}{5})$.

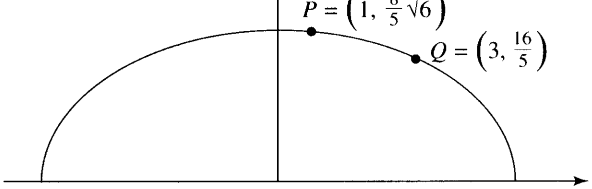


Figure 5.44

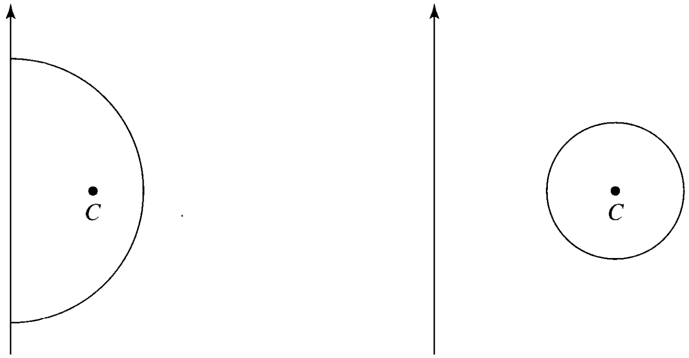


Figure 5.45

Don't forget to put in the limits of integration, but do not evaluate the integrals. In case (iii) use the fact from Exercise 9 that the slope of the tangent to the ellipse at any point (x,y) is $-\frac{4^2}{5^2}\frac{x}{y}$. Integral (iii) is a so-called "elliptic integral." It is difficult to solve.

35. Consider the hyperbola $\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$.

- **i.** Use Leibniz's tangent method to evaluate the slope m_P at any point $P = (x,y)$, $y \neq 0$, on the hyperbola.
- **ii.** Find a function whose graph is precisely the upper half of the hyperbola. Determine the derivative of this function.
- **iii.** Express as a definite integral (do not evaluate) the area under the upper half of this hyperbola and over the interval from 3 to 7.
- **iv.** Express as a definite integral (do not evaluate) the volume of the solid obtained by rotating the region of (iii) one complete revolution about the x-axis.
- **v.** Express as a definite integral (do not evaluate) the length of the hyperbolic arc from the point $(4, \frac{3}{2}\sqrt{7})$ to the point $(7, \frac{3}{2}\sqrt{40})$.

5I. Theorems of Pappus of Alexandria

Pappus of Alexandria (ca 300 A.D) was the last of the great Greek mathematicians. Theorems A and B that follow are named after him. Let S be either a curved segment in the plane or a region in the plane. In either case, think of S as made of a very thin homogeneous material. The *centroid* of S is its center of mass. For example, if the arc is a complete circle, the centroid is the center of the circle. The centroid of the region inside the circle is also the center of the circle.

A. If an arc that lies in a plane is revolved one complete revolution about an axis that lies in the plane but does not cross the arc, then the area of the surface that is formed is equal to the product of the length of the arc and the length of the path traced by the centroid of the arc. (To see what is going on, refer to the semicircular arc in Figure 5.45. The axis lies on the diameter of the circle. If the semicircular arc is allowed to rotate one complete revolution about this axis, the surface that is formed is a sphere. The centroid C of the arc will trace out a circle.)

B. If a region lying in a plane is revolved one complete revolution about an axis that lies in the plane but does not intersect the region, then the volume of the solid that is formed is equal to the product of the area of the region and the length of the path traced out by its centroid. (Again refer to Figure 5.45. The centroid of the circular disc on the right lies at the center C of the circle. If the disc is rotated one revolution about the axis, the volume formed is that of a donut. This is a "perfect" donut whose cross-sections are circles. The centroid C will trace out a circle.)

36. Use Pappus's theorem A to determine the area of the surface of a donut, given that the radius of its inner circle is r and that of its outer circle is R. (See Figure 5.46.)

37. Use Pappus's theorem A to determine the centroid of a semicircular arc. [Hint: Use the fact that the surface area of a sphere of radius r is $4\pi r^2$.]

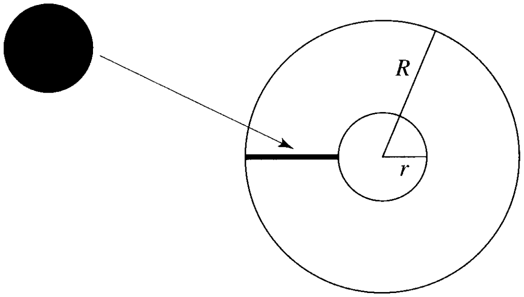


Figure 5.46

38. Use Pappus's theorem B to determine the centroid of a semicircular region. [Hint: Use the fact that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.]

Notes

[1] Historians of science recognized long ago that Leibniz and Newton invented the calculus independently. Newton did so in the years 1665 and 1666 but did not publish his findings until much later. Leibniz made his initial discoveries later, in the 1670s, but was first to publish them. Why did we start with Leibniz's version of the calculus instead of Newton's? Because this tactic provided an early opportunity to introduce Leibniz's superior notation. This is the notation that prevailed and is in essence the notation used today.

[2] Just so there is no misunderstanding, this title translates to *Six Exercises*.

[3] Like perhaps no other mathematician before him, Leibniz realized the fundamental importance of notation: "With the notation one has to consider the fact that it should be convenient as regards the process of invention. This is especially the case if it expresses the innermost nature of the thing with economy. In this way, the effort involved in the thought process is minimized in a wonderful way." To this end, he engaged in notational "experiments." For instance, he had originally written $\frac{d}{x}$ for dx and used omn. (from the Latin *omnes*, meaning "all") to denote $\int$. You will appreciate the importance of notation, if you try to multiply two large numbers written in Greek or Roman numerals.

[4] The development of the Fundamental Theorem of Calculus is based on Leibniz's article in the journal *Acta Eruditorum* in 1693, as translated into German (from the Latin) by Gerhard Kowalewski. See pages 24–34 of Band 162 of *Oswalds Klassiker der Exakten Wissenschaften*, Verlag von Wilhelm Engelmann, Leipzig, 1906. (The excerpts in English were provided by the author.) Other descriptions of Leibniz's work have made use of discussions in J. M. Child, *The Early Mathematical Manuscripts of Leibniz*, The Open Court Publishing Company, Chicago, London, 1920; and J. E. Hofmann, *Leibniz in Paris 1672–1676*, Cambridge University Press, Cambridge, 1974.

6

The Calculus of Newton

Isaac Newton (1642–1727) was born on Christmas day in Lincolnshire, England, the son of a farmer. He was educated at local schools and was a determined and successful student, but there was little to suggest that this quiet boy had exceptional promise. Since he had little inclination to be a farmer, his mother at last agreed with the schoolmaster to let him go off to the university. He passed his entrance examinations for Trinity College in Cambridge and began his studies there in 1661. He studied under Barrow and read Kepler's *Optics*, Euclid's *Elements* (which he at first found disappointingly simple), and works of Descartes and Galileo. In 1665, Newton took his degree of B.A. without any apparent distinction. Later in the year when the plague began to rage in London, Cambridge was closed as a precaution and he was forced to return to his birthplace in the country. Pondering alone with intense concentration, the twenty-three-year-old Newton hit upon the initial formulations of the ideas that would change many of the fundamental concepts of "natural philosophy," as science was referred to at that time. "I keep the subject of my inquiry constantly before me, and wait till the first dawning opens gradually, by little and little, into a full clear light," he wrote later. He had penetrating ideas about the nature of light, about the mathematical methods of a calculus of "fluents and fluxions," and about the motion of the heavenly bodies. "All this," he would write in his old age, "was in the two plague years of 1665 and 1666, for in those days I was in the prime of my age of invention, and minded mathematics and philosophy more than at any other time since." He became convinced that the motion of the planets around the Sun, as well as that of the Moon around the Earth, obeys the same physical laws as moving objects on Earth. A famous legend finds him meditating under an apple tree when, at the very moment that he heard the thud of a fallen apple, he was struck by the realization that the force that pulled the apple to Earth is one and the same as that which keeps the Moon in orbit around the Earth and from flying off into space.

Newton returned to Cambridge in 1667 and became absorbed with investigations into the nature of light. In experiments with prisms, he separated ordinary "white" light into its spectrum of colors from red to violet. The telescopes of the time relied on the principle of refraction. Lenses of appropriate shapes and configurations can be used to bend and redirect light rays in a way as to produce an enlarged image. (We will study the basic principles involved in Chapter 9.) However, the edges of a lense also act like prisms. As components of a refracting telescope, they create spectra of colors on the fringes of the field of vision that interfere with the image. Newton solved this problem by constructing a telescope in which the principal lens is replaced by a concave mirror, which redirects light by reflection rather than refraction. Newton's reflecting telescope had another advantage: it produced an image nine times larger than those of earlier refracting telescopes that were four times longer.

The description of the new telescope that was published by the *Royal Society of London for the Promotion of Natural Knowledge* generated such interest that Newton became famous overnight. The Royal Society was founded in 1660, and it was given a royal charter two years later. It counted the scientist Robert Hooke (1635–1703), the famous architect and mathematician Christopher Wren (1632–1723), and the astronomer Edmund Halley (1656–1742) among its members. It would play an important role throughout Newton's scientific life. In 1672, the Royal Society published Newton's first scientific work, which described his experiments on the nature of light. This article was very favorably received, but there was some criticism by Hooke and Huygens. Newton regarded these objections as deliberate efforts to annoy him: he published only very reluctantly thereafter.

6.1 Areas Under Simple Curves

In 1669 Newton first informed his colleagues about his mathematical discoveries by circulating the manuscript *De Analysi per Aequationes Numero Terminorum Infinitas*. It would not be published until 1711. Newton's teacher, Barrow, was so impressed by the vast abilities displayed in the "The Analysis by Means of Equations with an Infinite Number of Terms" that he resigned his professorship at Cambridge and arranged for Newton to succeed him. We shall now take a look at the study of areas under curves that the *De Analysi* undertakes. An early English translation[1] of this work (recall that the scientific language of the 17th century was Latin) begins as follows:

> The General Method which I had devised some considerable time ago for measuring the quantity of curves by an infinite series of terms you have, in the following, rather briefly explained than narrowly demonstrated.
>
> To the base *AB* of some curve *AD* let the coordinate *BD* be perpendicular [see Figure 6.1] and let *AB* be called x and *BD* y. Let again . . .

Newton next lets c be any constant, n and m integers, and he formulates his RULE I:

> If $cx^{\frac{m}{n}} = y$ then will $\frac{cn}{m+n}x^{\frac{m+n}{n}}$ equal the area *ABD*.

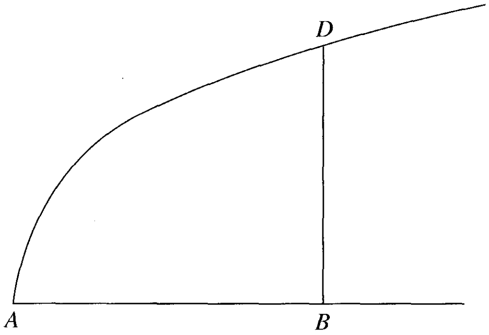


Figure 6.1

Without actually saying so, he assumes that $c > 0$, as well as $m > 0$ and $n > 0$. We will continue to follow Newton's argument closely, but we will use the language of functions, which clarifies what is going on. (Newton did not use functions explicitly.)

Denote the rational number $\frac{m}{n}$ by r. So $\frac{m+n}{n} = \frac{m}{n} + 1 = r + 1$. Now consider the function

$$f(x) = cx^r$$

Recall that $x^r = x^{\frac{m}{n}}$ is equal to $\sqrt[n]{x^m} = (x^m)^{\frac{1}{n}}$, i.e., it is the nth root of the mth power of x. Equivalently, $x^r = x^{\frac{m}{n}}$ is $(\sqrt[n]{x})^m = (x^{\frac{1}{n}})^m$, the mth power of the nth root of x. The functions

$$f(x) = 3x^{\frac{1}{2}} = 3\sqrt{x},\ f(x) = 6x^{\frac{3}{4}},$$

$$f(x) = 9x^{\frac{7}{5}},\ f(x) = 4x^2,\ \text{and}\ f(x) = 7x^3$$

are specific examples of the type of function that Newton is considering.

The general shape of the graph of $f(x) = cx^r$ depends on r. If $r < 1$, then the graph is the shaped like the one sketched by Newton; see Figure 6.2. The function $f(x) = 3x^{\frac{1}{2}} = 3\sqrt{x}$ is an example. If $r = 1$, then the graph of $f(x) = cx$ is the line through the origin with slope c. See Figure 6.3(a). Finally, if $r > 1$, then the general shape of the graph of $f(x) = cx^r$ is shown in Figure 6.3(b). The function $f(x) = 4x^2$ is a specific example. Recall from Section 4.3 that its graph is a parabola.

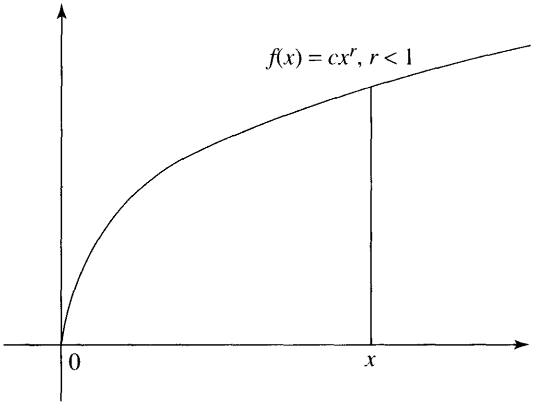
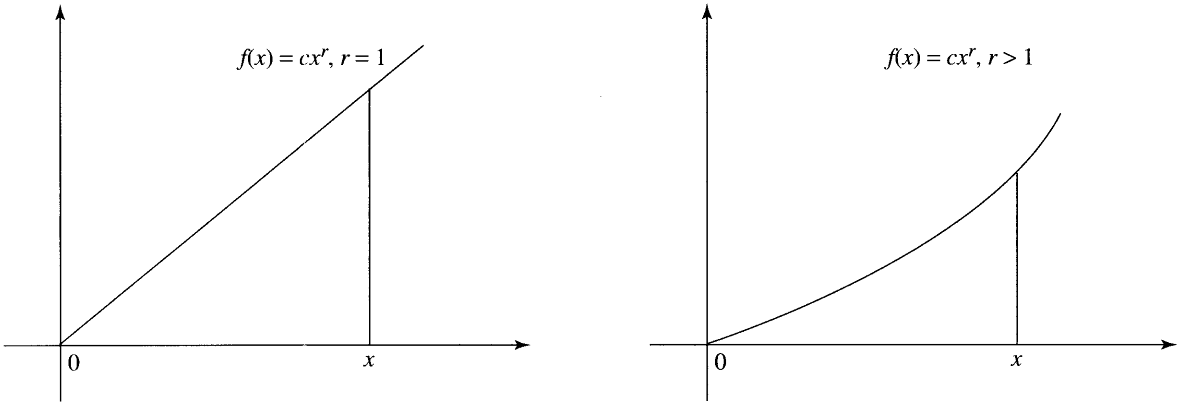


Figure 6.2

Focus on any function of the form $f(x) = cx^r$. We will assume with Newton that $x \geq 0$. Let $A(x)$ be the



(a)

(b)

Figure 6.3

area under the graph from 0 to x. This is the *area function* for the function f. Newton's first rule can now be restated as follows:

RULE I. If $f(x) = cx^r$, then its area function is $A(x) = \frac{c}{r+1}x^{r+1}$.

Example 6.1. Consider the function $f(x) = 3x^{\frac{1}{2}} = 3\sqrt{x}$. Since $r = \frac{1}{2}$ and $c = 3$, the area function is

$$A(x) = \frac{3}{\frac{3}{2}}x^{\frac{3}{2}} = 3 \cdot \frac{2}{3}x^{\frac{3}{2}} = 2x^{\frac{3}{2}}.$$

Taking $x = 5$, for example, we find that the area under the graph of $f(x) = 3x^{\frac{1}{2}}$ from 0 to 5 is $A(5) = 2 \cdot 5^{\frac{3}{2}} = 2(11.18) = 22.36$.

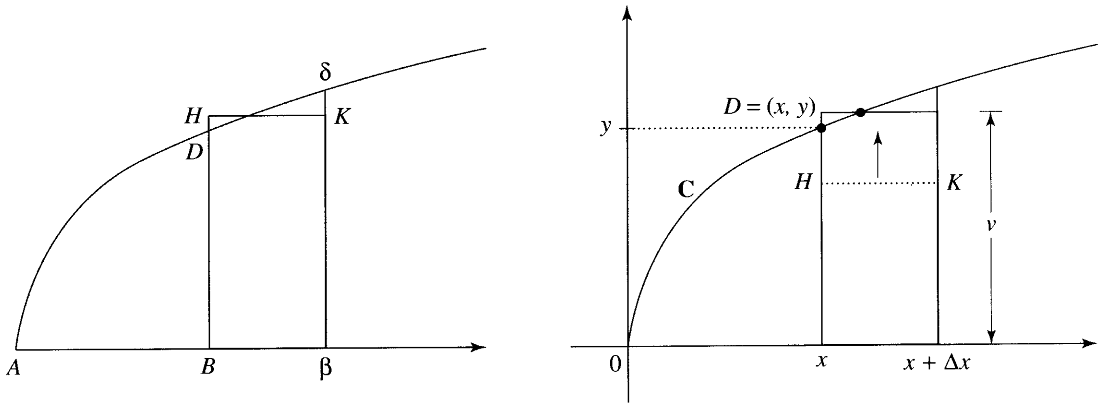
Example 6.2. Let $f(x) = 4x^2$. Now $r = 2$ and $c = 4$. So the area function is $A(x) = \frac{4}{3}x^3$. For instance, the area under the graph of $f(x) = 4x^2$ from 0 to 6 is $A(6) = \frac{4}{3} \cdot 6^3 = \frac{4}{3}(216) = 288$.

Let's now turn to Newton's proof, or verification, of his first rule. Figure 6.4(a) accompanies his argument. Newton "prepares" the proof by first considering a special case:

> *Preparation for demonstrating the first rule.* Let then $AD\delta$ be any curve whose base $AB = x$, the perpendicular ordinate $BD = y$, and the area $ABD = z$, as at the beginning. Likewise put $B\beta = o$, $\beta K = v$; and the rectangle $B\beta HK$ (ov) equal to the space $B\beta\delta D$. Therefore it is $A\beta = x + o$, and $A\delta\beta = z + ov$; which things being premised, assume any relation betwixt x and z that you please ...

This is the beginning of Newton's "preparation." We will follow Newton's argument closely. Consider a curve **C** as shown in Figure 6.4(b). Let x be any number on the horizontal axis with $x \geq 0$ and let $D = (x, y)$ be the corresponding point on C. In the discussion that follows, x and D are held fixed. As before, let $A(x)$ be the area under **C** from 0 to x. Now let Δx be a positive number and consider the point $x + \Delta x$. Note that Newton's z has been replaced by $A(x)$ and his o by Δx. These are the only changes in his original argument (other than the verbal aspects). Draw vertical lines through both x and $x + \Delta x$. Place the segment HK parallel to the horizontal axis; now raise it until the precise point is reached at which the area of the indicated rectangle is equal to the area under **C** from x to $x + \Delta x$. Let v be the y-coordinate of the point of intersection of HK and **C**. So v is the height of this rectangle. This is illustrated in Figure 6.4(b).

Continuing the preparation for his proof, Newton now assumes that the area under **C** from 0 to x is given



(a) **(b)**

Figure 6.4

by $A(x) = \frac{2}{3}x^{\frac{3}{2}}$ (for any $x \geq 0$) and asks the following questions: Does this assumption determine the curve **C**? If so, exactly what curve is it? His answers follow.

He observes that the area under the curve from 0 to $x + \Delta x$ is equal to $A(x) + \Delta x \cdot v$. Given his assumption about the area function, on the other hand, this area is also equal to $A(x + \Delta x) = \frac{2}{3}(x + \Delta x)^{\frac{3}{2}}$. So

$$A(x) + \Delta x \cdot v = A(x + \Delta x) = \frac{2}{3}(x + \Delta x)^{\frac{3}{2}}$$

Squaring both sides, he gets

$$\begin{aligned} &(A(x))^2 + 2A(x) \cdot \Delta x \cdot v + (\Delta x)^2 \cdot v^2 \\ &= \frac{4}{9}(x + \Delta x)^3 \\ &= \frac{4}{9}\left(x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3\right) \\ &= \frac{4}{9}x^3 + \frac{4}{3}x^2\Delta x + \frac{4}{3}x(\Delta x)^2 + \frac{4}{9}(\Delta x)^3. \end{aligned}$$

Since $(A(x))^2 = \frac{4}{9}x^3$, he obtains

$$2A(x) \cdot \Delta x \cdot v + (\Delta x)^2 \cdot v^2 = \frac{4}{3}x^2\Delta x + \frac{4}{3}x(\Delta x)^2 + \frac{4}{9}(\Delta x)^3$$

by canceling. Dividing both sides by Δx gives him

$$2A(x) \cdot v + \Delta x \cdot v^2 = \frac{4}{3}x^2 + \frac{4}{3}x\Delta x + \frac{4}{9}(\Delta x)^2.$$

Newton can now push Δx to zero on both sides. In the process, observe in Figure 6.4(b) that v closes in on y, and therefore that

$$2A(x)y = \frac{4}{3}x^2.$$

Since $A(x) = \frac{2}{3}x^{\frac{3}{2}}$, it follows that $\frac{4}{3}x^{\frac{3}{2}}y = \frac{4}{3}x^2$. So $y = \frac{x^2}{x^{\frac{3}{2}}}$. Therefore, the x- and y-coordinates of the typical point $D = (x, y)$ on the curve **C** satisfy the equation $y = x^{\frac{1}{2}}$. It follows that **C** is the graph of the function $f(x) = x^{\frac{1}{2}}$.

Newton's "preparation" has accomplished the following: He started with a curve **C** with area function given by $A(x) = \frac{2}{3}x^{\frac{3}{2}}$ and showed that **C** is the graph of the function $f(x) = x^{\frac{1}{2}}$. He could thus conclude that the area function $A(x)$ for the graph of $f(x) = x^{\frac{1}{2}}$ is $A(x) = \frac{2}{3}x^{\frac{3}{2}}$. This is precisely the assertion of his first rule for the case $c = 1$ and $r = \frac{1}{2}$.

Newton goes on to prove RULE I in its general form using the identical strategy. Instead of following Newton's argument directly, we will dissect it into its two main components. The first of these concerns derivatives.

Let f be any function given by a rule of the form $f(x) = kx^s$, where k is any constant and s is any

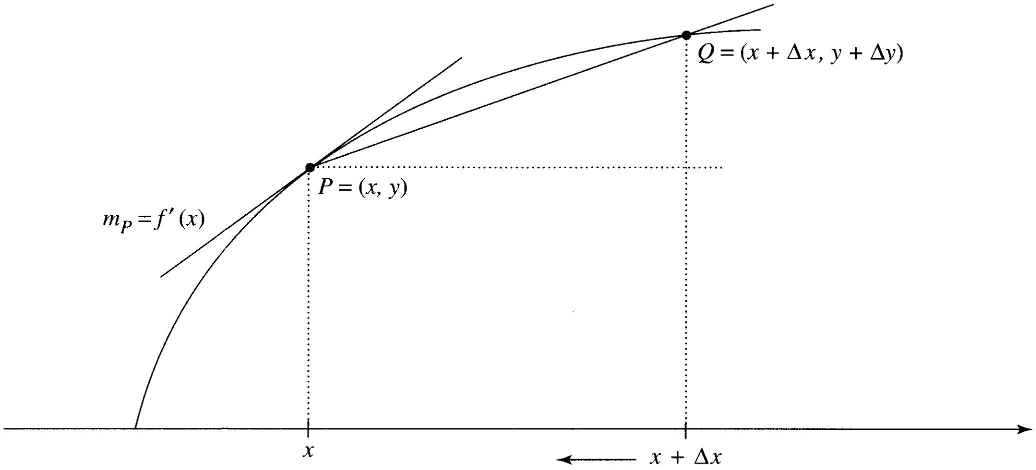


Figure 6.5

positive rational number. Newton needs to determine the derivative of f. He proceeds in the same way as Leibniz does in Section 5.2. (To be fair, he discovered this procedure before Leibniz.) He fixes any point $P = (x, y)$ on the graph of f; takes another point $Q = (x + \Delta x, y + \Delta y)$ on the graph; pushes Q to P and evaluates $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = m_P$. This is the slope of the tangent to the graph at $P = (x, y)$. Therefore $f'(x) = m_P$. See Figure 6.5.

Let's carry out the details. Put $s = \frac{m}{n}$ with m and n positive integers. Then the graph of f is the graph of the equation $y = kx^{\frac{m}{n}}$. By raising both sides to the nth power, $y^n = k^n x^m$. Because $Q = (x + \Delta x, y + \Delta y)$ is on the graph,

$$(y + \Delta y)^n = k^n (x + \Delta x)^m.$$

To multiply the n factors

$$(y + \Delta y)^n = (y + \Delta y)(y + \Delta y) \cdots (y + \Delta y)$$

out, both the y and Δy from any one group must be multiplied by the y and Δy from each of the other groups. The product $y \cdot y \cdots y = y^n$ is one term that arises in this way. Fixing any Δy and multiplying it by the y from each of the other $n - 1$ groups gives the product $y^{n-1}\Delta y$. Since n different Δy's can be picked to do this, $y^{n+1}\Delta y$ will occur n times. The result of the multiplication will therefore be

$$(y + \Delta y)^n = y^n + ny^{n-1}\Delta y + \ldots \text{ more terms.}$$

Notice that each of these additional terms must contain at least two factors of Δy (because the terms that contain no or one Δy are already accounted for). Doing the same thing with $(x + \Delta x)^m$, Newton arrives at an equation of the form

$$y^n + ny^{n-1}\Delta y + \text{terms with } (\Delta y)^2 \text{ as factor} =$$
$$k^n \Big(x^m + mx^{m-1}\Delta x + \text{terms with } (\Delta x)^2 \text{ as factor}\Big).$$

Because $P = (x, y)$ is on the graph, $y^n = k^n x^m$. Therefore, after a subtraction,

$$ny^{n-1}\Delta y + \text{ terms with } (\Delta y)^2 \text{ as factor} =$$
$$k^n \Big(mx^{m-1}\Delta x + \text{terms with } (\Delta x)^2 \text{ as factor}\Big).$$

Now divide both sides of this equation by Δx to get

$$ny^{n-1}\frac{\Delta y}{\Delta x} + \text{ terms with } \Delta y \cdot \frac{\Delta y}{\Delta x} \text{ as factor} =$$
$$k^n \Big(mx^{m-1} + \text{terms with } \Delta x \text{ as factor}\Big).$$

Finally, push Δx to zero. Since Δy also goes to 0 in the process (see Figure 6.5), all terms with $\Delta y \cdot \frac{\Delta y}{\Delta x}$ as factor go to zero, and because $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = m_P = f'(x)$, it follows that

$$(*) \qquad ny^{n-1}f'(x) = k^n m x^{m-1}.$$

Recalling that $y = kx^{\frac{m}{n}}$, we get by simple algebra that

$$y^{n-1} = k^{n-1}x^{\frac{m}{n}(n-1)} = k^{n-1}x^{m-\frac{m}{n}} = k^{n-1}x^m x^{-\frac{m}{n}}.$$

By substituting this for y^{n-1} in equation $(*)$ above, we obtain

$$nk^{n-1}x^m x^{-\frac{m}{n}} f'(x) = k^n m x^{m-1}.$$

After canceling, we have $nx \cdot x^{-\frac{m}{n}} f'(x) = km$. Since $\frac{m}{n} = s$, we get $nx^{1-s}f'(x) = km$. So

$$f'(x) = k\frac{m}{n} \cdot \frac{1}{x^{1-s}} = ksx^{s-1}.$$

Therefore, Newton has shown that

> The derivative of $f(x) = kx^s$ is $f'(x) = ksx^{s-1}$.

We will see later that this formula is valid for any rational number s, not only positive s. Taking $k = 1$, the formula asserts that the derivative of the function $f(x) = x^s$ is $f'(x) = sx^{s-1}$. Turn to Section 5.5A and observe that all of the cases of Example 5.16 follow from this fact.

Example 6.3. (a) Let $f(x) = 3x^{100}$. Then $f'(x) = 300x^{99}$. (b) If $g(x) = 2x^{-5} - 4x^{\frac{3}{5}}$, then $g'(x) = -10x^{-4} - \frac{12}{5}x^{-\frac{2}{5}}$.

The second component in Newton's verification of his RULE I is the Fundamental Theorem of Calculus. We already studied Leibniz's proof of this theorem in Section 5.4. We will describe Newton's proof next.

6.2 The Fundamental Theorem of Calculus (Again)

Consider any function f that satisfies $f(x) \geq 0$ for all x. For any $x \geq 0$, let $A(x)$ be the area under the graph from 0 to x. As was done earlier in Figure 6.4(b), let $x \geq 0$ be any number and Δx any distance, and construct the rectangle of height v in such a way that its area is equal to the area under the graph from x to $x+\Delta x$. Refer to Figure 6.6 and observe that

$$A(x + \Delta x) = A(x) + v\,\Delta x.$$

Therefore, $A(x + \Delta x) - A(x) = v\,\Delta x$, and hence

$$\frac{A(x + \Delta x) - A(x)}{\Delta x} = v.$$

Keeping x fixed, push Δx to zero, and observe that v gets pushed to $f(x)$ in the process. Thus

$$\lim_{\Delta x \to 0} \frac{A(x + \Delta x) - A(x)}{\Delta x} = f(x).$$

The left side is the derivative $A'(x)$ of $A(x)$. So Newton's argument has shown that the derivative $A'(x)$ of the area function $A(x)$ is the original function $f(x)$. Put another way:

> The area function A of a function f is an antiderivative of f.

What are the consequences of this fact? Refer to Figure 6.7 and observe that the area under the graph of f from a to b is equal to the area under the graph from 0 to b minus the area under the graph from 0 to a. So the area under the graph of f from a to b is equal to $A(b) - A(a)$. Recalling that Leibniz had written this area as $\int_a^b f(x)\,dx$, we see that

$$\int_a^b f(x)\,dx = A(b) - A(a).$$

Now let F be any antiderivative of f. Recall from Section 5.5B that any two antiderivatives of the same function differ only by a constant. Thus, $F(x) = A(x) + C$. So $F(b) - F(a) = (A(b) + C) - (A(a) + C) = A(b) - A(a)$. It follows that

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

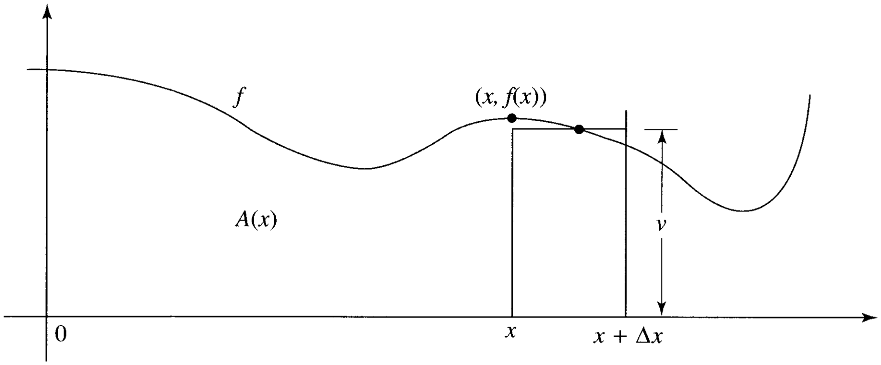


Figure 6.6

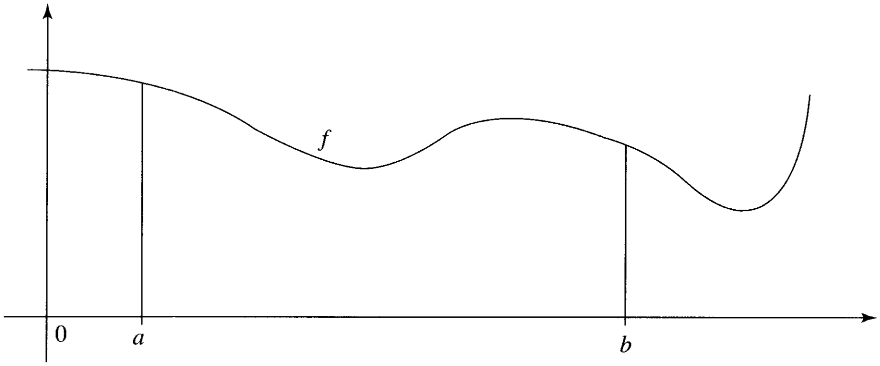


Figure 6.7

This is the Fundamental Theorem of Calculus already discussed in Chapter 5.

Newton's RULE I (slightly reformulated) is now easy. Let $F(x) = \frac{c}{r+1}x^{r+1}$. By the differentiation rule of Section 6.1, $F'(x) = \frac{c}{r+1}(r+1)x^r = cx^r$. So $F(x) = \frac{c}{r+1}x^{r+1}$ is an antiderivative of the function cx^r. Therefore, by the Fundamental Theorem of Calculus:

RULE I.

$$\int_a^b cx^r\,dx = \frac{c}{r+1}x^{r+1}\Big|_a^b = \frac{c}{r+1}b^{r+1} - \frac{c}{r+1}a^{r+1}.$$

Example 6.4. The area under the graph of $f(x) = 5x^3$ from $x = 1$ to $x = 4$ is equal to

$$\int_1^4 5x^3\,dx = \frac{5}{4}x^4\Big|_1^4 = \frac{5}{4}4^4 - \frac{5}{4}1^4 = 5 \cdot 4^3 - \frac{5}{4} = 318\frac{3}{4}.$$

Newton calls a function *simple* if it is of the form $f(x) = cx^r$, where $c > 0$ and $r > 0$. The theory he has developed allows him to find the area under the graph of a simple function. He next turns his attention to the area problem for functions obtained by adding terms of the form cx^r and he formulates his RULE II:

> If the value of y be made up of several such terms, the Area likewise shall be made up of

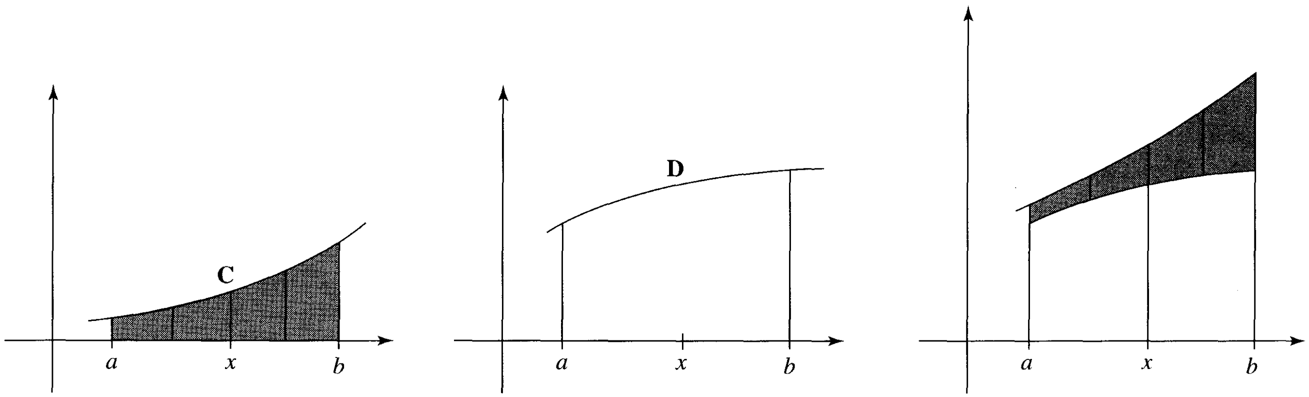


Figure 6.8

> the Areas which result from every one of the terms.

What he means is this. Let $f(x) = cx^r$ and $g(x) = dx^s$ be simple functions, and let **C** and **D** be their respective graphs. Then the area under the graph of the function $y = cx^r + dx^s$ from a to b is equal to the sum of the area under **C** from a to b plus the area under **D** from a to b. This is illustrated in Figure 6.8, where the graph of $y = cx^r + dx^s$ is the upper curve in the diagram on the right. The area underneath it is obtained by adding the shaded area (area under **C**) to the area under **D**.

In a more precise way, this addition rule is a consequence of the Fundamental Theorem of Calculus. Let f and g be functions with graphs above the x-axis. Let F and G be antiderivatives of f and g respectively. Then the areas under the respective graphs from a to b are given by

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

and

$$\int_a^b g(x)\,dx = G(b) - G(a).$$

By the sum rule for derivatives, $(F(x) + G(x))' = F'(x) + G'(x) = f(x) + g(x)$. So $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$. It follows that

$$\begin{aligned}\int_a^b (f(x) + g(x))\,dx &= (F(b) + G(b)) - (F(a) - G(a))\\ &= (F(b) - F(a)) + (G(b) - G(a))\\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.\end{aligned}$$

Therefore, Newton's second rule can be reformulated as follows:

RULE II.

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Example 6.5. Determine the area under the graph of $y = 4x^3 + 2x^{\frac{3}{2}}$ from 0 to 3.

Solution. Using RULE II in combination with RULE I, we get

$$\begin{aligned}\int_0^3 \left(4x^2 + 2x^{\frac{3}{2}}\right)\,dx &= \int_0^3 4x^2\,dx + \int_0^3 2x^{\frac{3}{2}}\,dx\\ &= \frac{4}{3}3^3 + \frac{4}{5}3^{\frac{5}{2}} = 36 + \frac{4}{5}(15.59)\\ &= 36 + 12.47 = 48.47.\end{aligned}$$

6.3 Computing Definite Integrals

Having dealt with simple functions and sums of simple functions, the *De Analysi* turns to the problem of determining areas under more complicated curves. Newton begins this discussion as follows:

> But if the value of y, or any of its terms be more compounded than the foregoing, it must be reduced into more simple terms; by performing the operation in letters, after the same manner as arithmeticians divide in decimal numbers, extract the square root, or resolve affected equations; and afterwards by the preceeding rules you will discover the Superficies of the curve sought.

What Newton has in mind is the reduction of the area problem for complicated functions to that for simple functions. But how? Consider, for example, the function $f(x) = \frac{1}{1+x}$. Here the "operation in letters" is polynomial division. Dividing $1+x$ into 1, Newton gets

$$\begin{array}{rl} & 1 \\ 1+x & \overline{)\,1} \\ \text{subtract} \longrightarrow & \underline{1+x} \\ & -x \end{array}$$

after the first step, and

$$\begin{array}{rl} & 1-x \\ 1+x & \overline{)\,1} \\ & \underline{1+x} \\ & -x \\ \text{subtract} \longrightarrow & \underline{-x-x^2} \\ & -x^2 \end{array}$$

after the second. Continuing in this way,

$$\begin{array}{rl} & 1-x+x^2-x^3+x^4 \\ 1+x & \overline{)\,1} \\ & \underline{1+x} \\ & -x \\ & \underline{-x-x^2} \\ & x^2 \\ & \underline{x^2+x^3} \\ & -x^3 \\ & \underline{-x^3-x^4} \\ & x^4 \\ & \underline{x^4+x^5} \\ & -x^5 \end{array}$$

This division process can be continued indefinitely.

If $|x| < 1$, the successive remainders

$$-x, x^2, -x^3, x^4, -x^5, \ldots, \pm x^i, \ldots$$

become smaller and smaller, so that $\frac{1}{1+x}$ is approximated by $1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$. The more terms that are included, the smaller the $\pm x^i$, and the better the approximation. This is similar to the way that the sum $1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \ldots$ approximates $\frac{4}{3}$ more and more accurately as more and more terms are added in. Refer to Section 3.3.

For example, let $x = 0.50$. Then $\frac{1}{1+x} = \frac{1}{1.5} = \frac{2}{3}$. Working with four-decimal accuracy (for instance) we get the following pyramid:

$$1-x = 1.000-0.5000 = 0.5000$$

$$1-x+x^2 = 0.5000+0.2500 = 0.7500$$

$$1-x+x^2-x^3 = 0.7500-0.1250 = 0.6250$$

$$1-x+x^2-x^3+x^4 = 0.6250+0.0625 = 0.6875$$

$$1-x+x^2-x^3+x^4-x^5 = 0.6875-0.0313 = 0.6562$$

$$1-x+x^2-x^3+x^4-x^5+x^6 = 0.6562+0.0156 = 0.6718$$

$$1-x+x^2-x^3+x^4-x^5+x^6-x^7 = 0.6718-0.0078 = 0.6640$$

$$1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8 = 0.6640+0.0039 = 0.6679$$

$$1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9 = 0.6679-0.0020 = 0.6659$$

$$1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+x^{10} = 0.6659+0.0010 = 0.6669$$

Observe that the numbers running down the pyramid on the right close in on $\frac{1}{1+x} = \frac{2}{3} = 0.6666\ldots$ as asserted. For a positive number x smaller than $0.5 = \frac{1}{2}$, the terms $\pm x^i$ will get smaller faster, and the convergence of

$$1 - x + x^2 - x^3 + x^4 - x^5 + \ldots \quad \text{to } \frac{1}{1+x}$$

will be more rapid. Take $i = 5$, for example. It follows from the pyramid that $\frac{1}{1+x}$ is approximated by the polynomial $1 - x + x^2 - x^3 + x^4 - x^5$ to within $0.6667 - 0.6562 = 0.0105$ over the entire interval from 0 to $\frac{1}{2}$.

Newton has therefore arrived at an approximation

$$\int_0^{\frac{1}{2}} \frac{1}{1+x}\,dx \approx \int_0^{\frac{1}{2}} (1 - x + x^2 - x^3 + x^4 - x^5)\,dx.$$

The definite integral on the right is easily evaluated by Newton's earlier analysis. In particular, taking antiderivatives term by term shows that

$$F(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6$$

is an antiderivative of $1 - x + x^2 - x^3 + x^4 - x^5$. Therefore, by the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{1}{1+x}\,dx &\approx \int_0^{\frac{1}{2}} (1 - x + x^2 - x^3 + x^4 - x^5)\,dx \\ &= F\left(\tfrac{1}{2}\right) - F(0) = F\left(\tfrac{1}{2}\right) \\ &= \tfrac{1}{2} - \tfrac{1}{2}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{3}\left(\tfrac{1}{2}\right)^3 - \tfrac{1}{4}\left(\tfrac{1}{2}\right)^4 + \tfrac{1}{5}\left(\tfrac{1}{2}\right)^5 - \tfrac{1}{6}\left(\tfrac{1}{2}\right)^6 \\ &\approx 0.5 - 0.125 + 0.042 - 0.016 + 0.007 - 0.003 \\ &= 0.405.\end{aligned}$$

If greater accuracy is required, we can use more terms in the approximation for $\frac{1}{1+x}$ and carry more decimals. Take, for instance,

$$1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - x^9 + x^{10}.$$

Continuing the pattern of the preceeding computation for five more steps, we get

$$\begin{aligned}&\int_0^{\frac{1}{2}} \frac{1}{1+x}\,dx \\ &\quad \approx \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 - \ldots + \frac{1}{11}\left(\frac{1}{2}\right)^{11}.\end{aligned}$$

Round off at the sixth decimal place and show that this sum is equal to

$$\begin{aligned}&0.500000 - 0.125000 + 0.041667 - 0.015625 \\ &\quad + 0.006250 - 0.002604 + 0.001116 - 0.000488 \\ &\quad + 0.000217 - 0.000098 + 0.000044 \\ &= 0.405479.\end{aligned}$$

To evaluate $\int_0^{\frac{1}{2}} \frac{1}{1+x}\,dx$ with an accuracy of up to, say, four decimal places, use the following rule of thumb. Keep computing (and adding and subtracting terms) until the term is reached that is zero when rounded to four-decimal places. The approximation obtained at this point should be accurate up to four decimal places. In this example, the term that rounds to zero is 0.000044. So the approximation 0.405479 should be accurate up to four decimals. This is correct since the value of the integral $\int_0^{\frac{1}{2}} \frac{1}{1+x}\,dx$ up to six-decimal-place accuracy is 0.405465. (The determination of the precise value of this integral makes use of logarithms and will be discussed in Chapter 10.) A word of caution is in order. This rule of thumb is not always accurate; this is due to the phenomenon known as *roundoff error*.

Observe that $\int_0^{\frac{1}{2}} \frac{1}{1+x}\,dx$ represents the area under the graph of $f(x) = \frac{1}{1+x}$ between $x = 0$ and $x = \frac{1}{2}$. See Figure 6.9. So up to four-decimal accuracy, this area is equal to 0.4055.

Newton also derives the approximation

$$\begin{aligned}&\sqrt{1+x} = (1+x)^{\frac{1}{2}} \\ &\qquad \approx 1 + \binom{\frac{1}{2}}{1}x + \binom{\frac{1}{2}}{2}x^2 + \binom{\frac{1}{2}}{3}x^3\end{aligned}$$

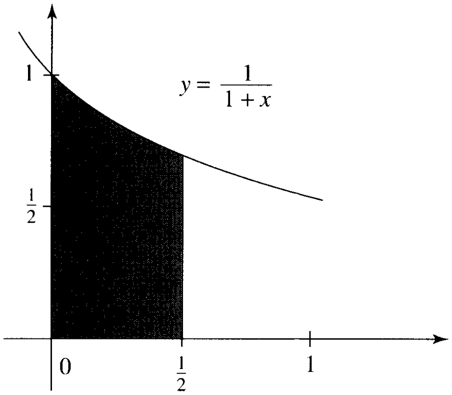


Figure 6.9

$$+\binom{\frac{1}{2}}{4}x^4 + \cdots + \binom{\frac{1}{2}}{i}x^i + \ldots,$$

where

$$\binom{\frac{1}{2}}{1} = \frac{1}{2},\ \binom{\frac{1}{2}}{2} = \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2!},\ \binom{\frac{1}{2}}{3} = \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}$$

and in general,

$$\binom{\frac{1}{2}}{i} = \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)\ldots(\frac{1}{2}-(i-1))}{i!},$$

for any positive integer i. Recall that

$$i! = i\cdot(i-1)\cdot(i-2)\cdots 3\cdot 2\cdot 1$$

for any integer $i > 0$. So $3! = 3\cdot 2\cdot 1 = 6$; $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$; and so on. Computing the first few terms shows that

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \ldots.$$

This approximation works for any x with $|x| \le 1$. Again, the inclusion of more terms increases the accuracy.

To get a feeling for the validity of this approximation for $\sqrt{1+x}$ let's consider the first five terms, $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$. If Newton is right, then the square

$$\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4\right)$$

$$\cdot\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4\right)$$

of this polynomial should be a quantity "close to" $1+x$. Carry out this product as follows: Multiply the 1 on the top by all the terms on the bottom, then the $\frac{1}{2}x$ on the top by all the terms on the bottom, and so on. This results in

$$\begin{array}{l}
1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3-\tfrac{5}{128}x^4\\
\quad+\tfrac{1}{2}x+\tfrac{1}{2}\cdot\tfrac{1}{2}x^2-\tfrac{1}{2}\cdot\tfrac{1}{8}x^3+\tfrac{1}{2}\cdot\tfrac{1}{16}x^4-\tfrac{1}{2}\cdot\tfrac{5}{128}x^5\\
\qquad-\tfrac{1}{8}x^2-\tfrac{1}{8}\cdot\tfrac{1}{2}x^3+\tfrac{1}{8}\cdot\tfrac{1}{8}x^4-\tfrac{1}{8}\cdot\tfrac{1}{16}x^5+\tfrac{1}{8}\cdot\tfrac{5}{128}x^6\\
\qquad\quad+\tfrac{1}{16}x^3+\tfrac{1}{16}\cdot\tfrac{1}{2}x^4-\tfrac{1}{16}\cdot\tfrac{1}{8}x^5+\tfrac{1}{16}\cdot\tfrac{1}{16}x^6-\tfrac{1}{16}\cdot\tfrac{5}{128}x^7\\
\qquad\qquad-\tfrac{5}{128}x^4-\tfrac{5}{128}\cdot\tfrac{1}{2}x^5+\tfrac{5}{128}\cdot\tfrac{1}{8}x^6-\tfrac{5}{128}\cdot\tfrac{1}{16}x^7+\tfrac{5}{128}\cdot\tfrac{5}{128}x^8
\end{array}$$

After adding terms column by column and computing the coefficients, this becomes

$$1 + x + \left(-\frac{14}{256}x^5 + \frac{14}{1024}x^6 - \frac{10}{2048}x^7 + \frac{25}{16{,}384}x^8\right).$$

A moment's thought reveals that for $-1 \le x \le 1$, the largest value of

$$-\frac{14}{256}x^5 + \frac{14}{1024}x^6 - \frac{10}{2048}x^7 + \frac{25}{16{,}384}x^8$$

occurs when $x = -1$. This value is $\frac{896+224+80+25}{16{,}384} = \frac{1225}{16{,}384} = 0.075$. So over the interval $-1 \le x \le 1$,

$$1 + x \approx \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4\right)^2$$

to within 0.075. Taking square roots, gives

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4.$$

As already remarked, approximations such as this can be made as accurate as required by adding on more terms.

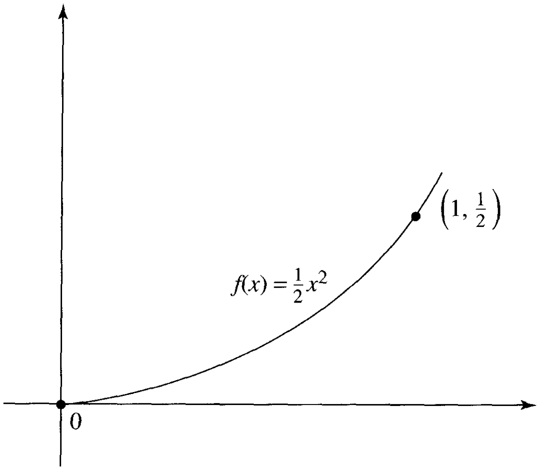


Figure 6.10

Example 6.6. Compute the length of the arc of the parabola $y = \frac{1}{2}x^2$ from $(0, 0)$ to $(1, \frac{1}{2})$. Let $f(x) = \frac{1}{2}x^2$; then $f'(x) = x$ and $(f'(x))^2 = x^2$. See Figure 6.10. The length of the arc is provided by the definite integral in Section 5.6C:

$$\int_0^1 \sqrt{1 + (f'(x))^2}\,dx = \int_0^1 \sqrt{1 + x^2}\,dx$$

How can we compute it? Replacing x by x^2 in the approximation

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$$

provides the approximation

$$\sqrt{1+x^2} \approx 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \frac{5}{128}x^8.$$

Taking antiderivatives term by term shows that

$$F(x) = x + \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{112}x^7 - \frac{5}{1152}x^9$$

is an antiderivative of $1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \frac{5}{128}x^8$. Therefore,

$$\begin{aligned}
&\int_0^1 \sqrt{1+x^2}\,dx \\
&\approx \int_0^1 \left(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \frac{5}{128}x^8\right) dx \\
&= \left(x + \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{112}x^7 - \frac{5}{1152}x^9\right)\Bigg|_0^1 \\
&= 1 + \frac{1}{6} - \frac{1}{40} + \frac{1}{112} - \frac{5}{1152} \\
&= 1 + 0.1667 - 0.0250 + 0.0089 - 0.0043 \\
&= 1.1466.
\end{aligned}$$

Again, this approximation can be made accurate to any desired decimal place by continuing the pattern and taking enough terms.

In addition to the approximations

$$\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$$

and

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots,$$

Newton also developed the trigonometric approximations

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

and

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots .$$

Such "infinite" polynomials are called *power series* in modern mathematics.[2] Power series have remained in a central position in mathematics ever since Newton first made use of them.

We have given an illustration of Newton's power series method for computing definite integrals. Definite integrals can represent areas, volumes, lengths of arcs, indeed—as we will see in subsequent chapters—they can represent many other physical quantities as well. Newton, therefore, produced a powerful method of calculation that is applicable to a large variety of mathematical problems.

6.4 Moving Points

In 1671 Newton gave a second, more extensive, exposition of his ideas in the treatise *Methodus Fluxionum et Series Infinitarum* (Method of Fluxions and Infinite Series). It would not be published until 1736. After discussing the role of power series as a tool for computing areas under curves, Newton goes on to say

> So much for computational methods of which in the sequel I shall make frequent use. It now remains, in illustration of this analytic art, to deliver some typical problems and such especially as the nature of curves represent. But first of all I would observe that difficulties of this sort may all be reduced to these two problems alone, which I may be permitted to propose with regard to the space traversed by any local motion however accelerated or retarded:
>
> **(1)** Given the length of the space continuously (that is at every time), to find the speed of motion at any time proposed.
>
> **(2)** Given the speed of motion continuously, to find the length of the space described at any time proposed.

What is Newton saying here? Consider a point or a particle moving in the Cartesian plane. Take a stopwatch and let it start at time $t = 0$. After any time $t \geq 0$ it will have some position in the plane. Let the x-coordinate of this position be $x(t)$ and the y-coordinate $y(t)$. Both $x(t)$ and $y(t)$ are functions of t. Newton calls such functions *fluents*. The position of the point and hence the coordinates $x(t)$ and $y(t)$ will vary with t. Figure 6.11 shows a typical situation. At $(x(t), y(t))$ the point will have a certain velocity in the x-direction and also in the y-direction. Newton calls these velocities the *fluxions* of $x(t)$ and $y(t)$. These velocities are also functions of t. The two problems that Newton has singled out are these:

(1) Given that the position of the point is known at any time, determine its velocity at any time. For example, given that the function $x(t)$ is known explicitly, determine the function that gives the velocity in the x-direction.

(2) Given that the velocity of the point is known at any time, determine its position at any time. For example, given the function that describes the velocity in the y-direction, determine the function $y(t)$.

The primary objective of Newton's abstract discussion is the development of the mathematics that underlies the phenomenon of motion. This is precisely what is necessary for a fundamental understanding of the basic observations of Galileo that we discussed in Section 4.2. Newton's mathematics of motion will be the focus of the rest of the chapter. In reference to Figure 6.11, observe that if the functions $x(t)$ and $y(t)$ are both understood, then the motion of the point is understood. The mathematics of motion in the plane therefore reduces to the situation of a point moving along a coordinatized line.

Suppose a point is moving along a coordinatized line. Begin to observe it at time $t = 0$. Denote its coordinate at any time $t \geq 0$ by $p(t)$. Note that p is a function of time t. It is the *position function* of the point. (See Figure 6.12.) The position $p(0)$ is the *initial position* of the point. The units of distance and time can be miles and hours, meters and seconds, etc., as required in a specific problem.

Example 6.7. Suppose that p is given explicitly by $p(t) = t$. So the position of the point at time $t = 0$ is $p(0) = 0$; at time $t = 1$, it is $p(1) = 1$; at time $t = 2$, it is $p(2) = 2$, and so on. Next suppose that the position function is $p(t) = t^2$. Then $p(0) = 0$, $p(1) = 1$, $p(2) = 4$, and so forth. The motion of the point is shown in Figure 6.13(a) in the first case and in Figure 6.13(b) in the second. In each case, the location of the point is listed above the axis and the time at which the point is at a given location is indicated below the axis.

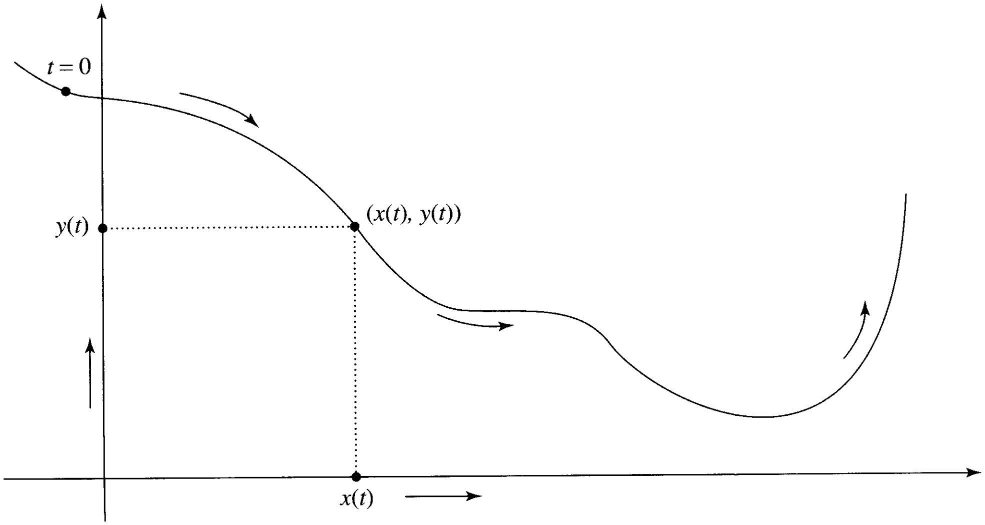


Figure 6.11

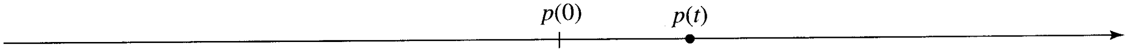


Figure 6.12

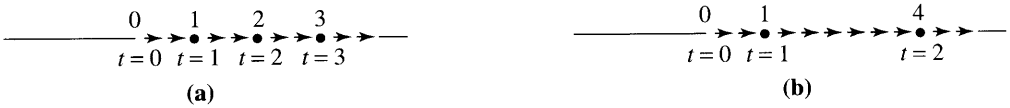


Figure 6.13

Example 6.8. Suppose that $p(t) = t^2 - 4t + 3$. Before considering the motion of a point with position function p, we will investigate the graph of p (with horizontal axis the t-axis). Note that the graph is a parabola that opens upward. The derivative $p'(t) = 2t - 4$ is 0 when $t = 2$. Since $p(2) = 4 - 8 + 3 = -1$, it follows that the parabola has a horizontal tangent at $(2, -1)$. This point is at the bottom of the parabola, so the smallest value that $p(t)$ can have is -1 and it occurs when $t = 2$. Since $t^2 - 4t + 3 = (t - 1)(t - 3)$, it follows that $p(t) = 0$ precisely when $t = 1$ or $t = 3$. The graph of p is sketched in Figure 6.14. Be aware of the fact that it is the graph of the position function; it is not the trajectory of the moving point.

We now turn to a point that moves on a horizontal axis with position given by

$$p(t) = t^2 - 4t + 3 = (t - 1)(t - 3)$$

at any time $t \geq 0$. Since $p(0) = 3$, the point is at 3 on the axis at $t = 0$. Since $p(t)$ decreases initially (the graph in Figure 6.14 falls), the point moves to the left from this position and reaches the point 0 at $t = 1$ and the point -1 at $t = 2$. There it stops, changes direction (the graph begins to rise), and at $t = 3$ it is back at 0. At $t = 4$ it is at 3 and continues moving to the right.

The motion of the point is described schematically in Figure 6.15. The points singled out on the axis are -1, 0, and 3.

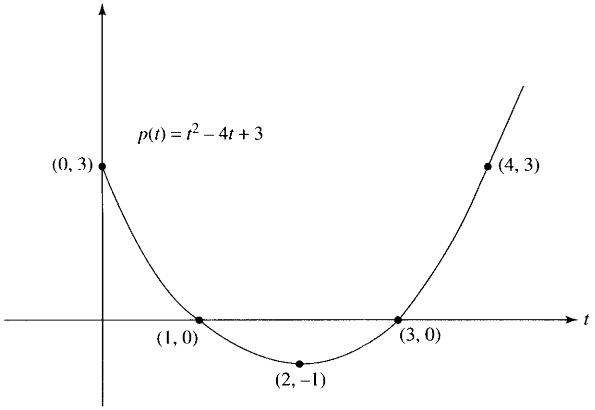


Figure 6.14

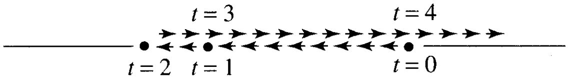


Figure 6.15

Let's return to the general case of a point moving on a coordinatized axis. Fix some instant $t_1 \geq 0$. Let an additional amount of time Δt elapse and consider $t_1 + \Delta t$. We will denote the time interval from t_1 to $t_1 + \Delta t$ by $[t_1, t_1 + \Delta t]$. We have

$$\begin{aligned}
p(t_1) &= \text{the position at time } t_1 \\
p(t_1 + \Delta t) &= \text{the position at time } t_1 + \Delta t \\
p(t_1 + \Delta t) - p(t_1) &= \text{the change in position over the time interval } [t_1, t_1 + \Delta t] \\
&= \text{the net distance (possibly negative) covered by the point in the time interval } [t_1, t_1 + \Delta t] \\
\frac{p(t_1 + \Delta t) - p(t_1)}{\Delta t} &= \text{the net distance covered divided by the time it took to cover that distance} \\
&= \text{the average velocity during the time interval } [t_1, t_1 + \Delta t]
\end{aligned}$$

Example 6.9. Return to the point with position function $p(t) = t^2 - 4t + 3 = (t-1)(t-3)$ for $t \geq 0$. To make things more concrete, suppose that distance is measured in inches and time in seconds. Take $t_1 = 1$ second. Now the point is exactly one second into its trip. Let another $\Delta t = 3$ seconds elapse. So $t_1 + \Delta t = 4$ seconds. Since $p(t_1 + \Delta t) - p(t_1) = p(4) - p(1) = 3 - 0 = 3$, the point has covered a net distance of 3 inches in these 3 seconds. Because

$$\frac{p(t_1 + \Delta t) - p(t_1)}{\Delta t} = \frac{p(4) - p(1)}{3} = \frac{3 - 0}{3} = 1,$$

the point has an average velocity of 1 inch per second over the time interval $[t_1, t_1 + \Delta t] = [1, 4]$. Next take $\Delta t = 2$. Now $t_1 + \Delta t = 3$, and

$$\frac{p(t_1 + \Delta t) - p(t_1)}{\Delta t} = \frac{p(3) - p(1)}{2} = \frac{0 - 0}{2} = 0.$$

The net distance covered by the point is zero and it has an average velocity of zero. With $\Delta t = 1$, we have $t_1 + \Delta t = 2$, and the average velocity of

$$\frac{p(2) - p(1)}{1} = \frac{-1 - 0}{1} = -1 \text{ inches per second.}$$

The minus sign indicates that the point has moved in the negative direction (from $p(1) = 0$ to $p(2) = -1$) over the time interval $[t_1, t_1 + \Delta t] = [1, 2]$.

For any moving point, consider the ratio

$$\frac{p(t_1 + \Delta t) - p(t_1)}{\Delta t}$$

for progressively smaller elapsed times Δt. This is the average velocity of the point computed over smaller and smaller time frames $[t_1, t_1 + \Delta t]$ near t_1. (See Figure 6.16.) In the process, $\frac{p(t_1 + \Delta t) - p(t_1)}{\Delta t}$ closes in on a number called the *velocity at the instant* t_1; it is denoted by $v(t_1)$. This is the rate at which the position changes at the instant t_1. It is obtained by pushing

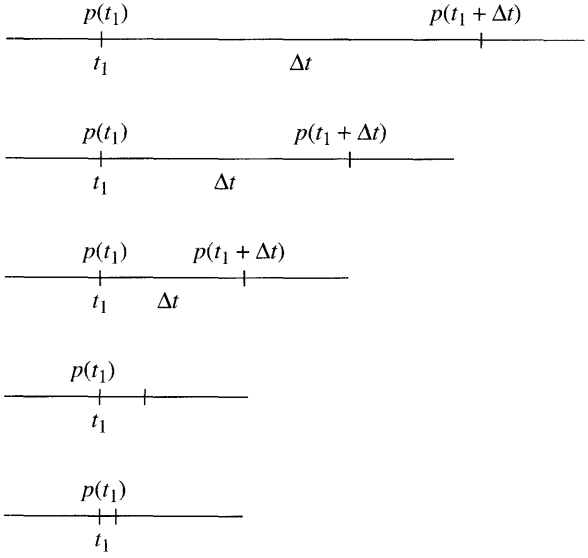


Figure 6.16

Δt to zero and observing what happens to the ratio $\frac{p(t_1+\Delta t)-p(t_1)}{\Delta t}$. Therefore,

$$v(t_1) = \lim_{\Delta t \to 0} \frac{p(t_1 + \Delta t) - p(t_1)}{\Delta t}.$$

The velocity $v(0)$ at $t = 0$ is called the *initial velocity* of the point. It is possible for Δt to be negative. This means that $t_1 + \Delta t$ occurs before t_1 and that the time interval in question is $[t_1 + \Delta t, t_1]$.

Example 6.10. Let's see what happens when this is done with $t_1 = 1$ second for the point with position $p(t) = t^2 - 4t + 3 = (t - 1)(t - 3)$ for $t \geq 0$. Taking $\Delta t = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{64}$, we get

$$t_1 + \Delta t = \frac{3}{2}, \frac{5}{4}, \frac{9}{8}, \ldots, \frac{65}{64}.$$

The successive average velocities are

$$\frac{p\left(\frac{3}{2}\right) - p(1)}{\frac{1}{2}} = \frac{\left(\frac{3}{2} - 1\right) \cdot \left(\frac{3}{2} - 3\right) - 0}{\frac{1}{2}} = \frac{1}{2}\left(-\frac{3}{2}\right)2$$

$$= -1.5 \text{ inches per second,}$$

$$\frac{p\left(\frac{5}{4}\right) - p(1)}{\frac{1}{4}} = \frac{\left(\frac{5}{4} - 1\right) \cdot \left(\frac{5}{4} - 3\right) - 0}{\frac{1}{4}} = \frac{1}{4}\left(-\frac{7}{4}\right)4$$

$$= -1.75 \text{ inches per second,}$$

$$\frac{p\left(\frac{9}{8}\right) - p(1)}{\frac{1}{8}} = \frac{\left(\frac{9}{8} - 1\right) \cdot \left(\frac{9}{8} - 3\right) - 0}{\frac{1}{8}} = \frac{1}{8}\left(-\frac{15}{8}\right)8$$

$$= -1.88 \text{ inches per second,} \ldots,$$

$$\frac{p\left(\frac{65}{64}\right) - p(1)}{\frac{1}{8}} = \frac{\left(\frac{65}{64} - 1\right) \cdot \left(\frac{65}{64} - 3\right) - 0}{\frac{1}{64}} = \frac{1}{64}\left(-\frac{127}{64}\right)64$$

$$= -1.98 \text{ inches per second.}$$

Continuing to push Δt to zero, we see that

$$v(1) = \lim_{\Delta t \to 0} \frac{p(1 + \Delta t) - p(1)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta t(\Delta t - 2) - 0}{\Delta t}$$

$$= \lim_{\Delta t \to 0} (\Delta t - 2) = -2 \text{ inches per second.}$$

Again, the minus sign means that the point is moving in the negative direction at time $t_1 = 1$. Look again at Figure 6.15.

Refer back to Section 5.5A and observe that the limit

$$\lim_{\Delta t \to 0} \frac{p(t_1 + \Delta t) - p(t_1)}{\Delta t}$$

has three different meanings simultaneously:

(1) It is the velocity $v(t_1)$ at time t_1 of the point whose position function is p;

(2) It is the value $p'(t_1)$ of the derivative of the position function p;

(3) It is the slope of the tangent line of the graph of the function p at the point $(t_1, p(t_1))$.

To repeat, if p is the position function of a moving point, then its derivative

$$p'(t_1) = \lim_{\Delta t \to 0} \frac{p(t_1 + \Delta t) - p(t_1)}{\Delta t}$$

is the velocity $v(t_1)$ of the point at time t_1. Changing notation from t_1 to t, we see that $p'(t)$ is the velocity $v(t)$ of the point at any time t. If $v(t)$ is positive, then the point is moving to the right at time t, and if $v(t)$ is negative, the point is moving to the left at time t. The *speed* of the point at time t is the absolute value $|v(t)|$

of the velocity. Note that velocity is the combination of speed and direction. The concepts velocity and speed are often used interchangeably in this book when the direction of the motion is clear from the context.

Example 6.11. The point in Example 6.10 has a velocity of $v(t) = p'(t) = 2t - 4 = 2(t-2)$ and a speed of $|2t - 4|$ at any time t. Since $v(t)$ is negative for $0 \le t < 2$, the point moves to the left during this time; since $v(2) = 0$, the point stops at $t = 2$; and since $v(t)$ is positive for $t > 2$, it moves to the right for any such t. Compare this analysis with the information in Figure 6.15. When $t = 1$, the point's velocity is -2 inches per second (as already observed) and its speed is $|-2| = 2$ inches per second.

Let's see whether the definition of the velocity as the derivative of the position function agrees with what we know about velocity intuitively. Consider a point moving on the line and let its position be $p(t)$ at any time $t \ge 0$. Suppose that after some time t_1, when its position is $p(t_1)$, it continues with the constant velocity $p'(t_1)$. This means that $p'(t) = p'(t_1)$ for any $t \ge t_1$. So for $t \ge t_1$, the graph of $p(t)$ has constant slope $p'(t_1)$. Since the graph of $p(t)$ goes through the point $(t_1, p(t_1))$, we find from the point-slope form of the equation of a line that

$$p(t) - p(t_1) = p'(t_1)(t - t_1),$$

or

$$p(t) = p(t_1) + p'(t_1)(t - t_1),$$

for $t \ge t_1$. For an elapsed time Δt after t_1, the point will have moved a distance of

$$\begin{aligned} &p(t_1 + \Delta t) - p(t_1) \\ &\quad = p(t_1) + p'(t_1)\big((t_1 + \Delta t) - t_1\big) - p(t_1) \\ &\quad = p'(t_1)\,\Delta t. \end{aligned}$$

But since distance = velocity × time when velocity is constant, this is exactly what we would expect. Figure 6.17 shows what is going on from the point of view of the graph of the position function p.

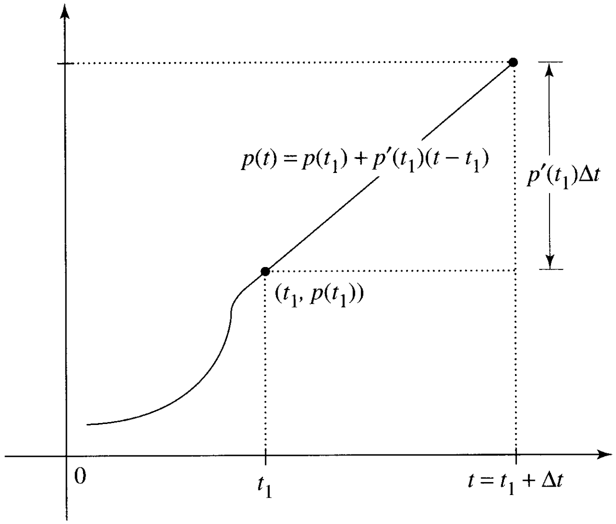


Figure 6.17

Now let's focus on the velocity function $v(t)$ of a point moving on a coordinatized axis. See Figure 6.12. Fix some instant $t_1 \ge 0$. Let an additional amount of time Δt elapse and consider $t_1 + \Delta t$. Then

$$\begin{aligned} v(t_1) &= \text{the velocity at time } t_1 \\ v(t_1 + \Delta t) &= \text{the velocity at time } t_1 + \Delta t \\ v(t_1 + \Delta t) - v(t_1) &= \text{the change in velocity over the} \\ &\ \text{time interval } [t_1, t_1 + \Delta t] \\ \frac{v(t_1 + \Delta t) - v(t_1)}{\Delta t} &= \text{the average change in velocity} \\ &\ \text{per unit time over } [t_1, t_1 + \Delta t] \\ &= \text{the average acceleration} \\ &\ \text{over } [t_1, t_1 + \Delta t] \end{aligned}$$

Consider the average acceleration $\frac{v(t_1+\Delta t)-v(t_1)}{\Delta t}$ over the time interval $[t_1, t_1 + \Delta t]$ for progressively smaller elapsed times Δt. In the process, the ratio $\frac{v(t_1+\Delta t)-v(t_1)}{\Delta t}$ closes in on the number called the *acceleration at the instant* t_1. It is denoted by $a(t_1)$ and is the rate at which the velocity is changing at the instant t_1. In mathematical shorthand, the acceleration at time

t_1 is

$$a(t_1) = \lim_{\Delta t \to 0} \frac{v(t_1 + \Delta t) - v(t_1)}{\Delta t},$$

Changing notation from t_1 to t, we find that $a(t)$ is the derivative $v'(t)$ of the velocity function $v(t)$. Since $v(t) = p'(t)$, $a(t)$ is the derivative of $p'(t)$. This *second derivative* of p is denoted $p''(t)$. So $a(t) = p''(t)$. What units is acceleration measured in? For example, if distance is given in inches and time in seconds, then velocity is given in inches per second, and acceleration in (inches per second) per second, or inches per second^2.

Example 6.12. Let's have one more look at the point with position function $p(t) = t^2 - 4t + 3$. Since $v(t) = 2t - 4$, the point's acceleration is $a(t) = v'(t) = 2$. Suppose that the point is a particle of mass m. By one of Newton's laws of motion, an acceleration a on a body of mass m is the consequence of the action of a force F. The connection between the force and the acceleration is given by the famous equation $F = ma$. So the force on this particle is $F = 2m$. It is positive and hence "pushes" the particle in the positive direction. The motion of the particle has the following interpretation. It starts at time $t = 0$ at $p(0) = 3$ with an initial velocity of $v(0) = -4$. So it begins by moving to the left. Since the force acts to the right, the particle must slow down. It eventually comes to a halt. Since $v(2) = 0$, this happens at time $t = 2$. From this time onward, the force accelerates the particle to the right. Check this against Figure 6.15.

We have analyzed the motion of a point on a line and observed how its position function determines both its velocity and acceleration. We will now see that this reasoning can be reversed. Given the acceleration of a point, it is possible to determine first its velocity and then its position. This will make important use of a fact from Section 5.5B: If two functions have the same derivative, then one of them is equal to the other plus a constant. In particular, if the derivative of a function is always zero, the function is a constant.

Example 6.13. Let a point move on an axis with acceleration $a(t) = v'(t) = p''(t) = 0$ for all $t \geq 0$. By the fact just referred to, this means that $v(t) = C$, for some constant C. So $v(t) = C = v(0)$. Notice that the derivatives of both $v(0) \cdot t$ and the position function are equal to $v(0)$. Therefore, $p(t) = v(0)t + D$ for some constant D. Taking $t = 0$ shows that $D = p(0)$ is the initial position of the point. The position function is now explicitly given by

$$p(t) = v(0)t + p(0).$$

Example 6.14. Suppose a point moves along an axis with constant acceleration $a(t) = v'(t) = p''(t) = k$ for all $t \geq 0$. Observe that $v(t)$ and kt both have the derivative k. It follows that $v(t) = kt + C$. Setting $t = 0$, we see that $C = v(0)$ is the initial velocity; hence

$$v(t) = kt + v(0).$$

Since the derivative of $\frac{k}{2}t^2 + v(0)t$ is equal to

$$kt + v(0) = v(t),$$

it follows that $\frac{k}{2}t^2 + v(0)t$ and the position function $p(t)$ have the same derivative. Hence $p(t) = \frac{k}{2}t^2 + v(0)t + D$. Plugging in $t = 0$ shows that $D = p(0)$ is the initial position of the point. So the position function is equal to

$$p(t) = \frac{k}{2}t^2 + v(0)t + p(0).$$

If the initial position is $p(0) = 0$ and the initial velocity $v(0) = 0$, then $p(t) = \frac{k}{2}t^2$. In this case, the constant acceleration k produces a displacement of $\frac{k}{2}t^2$ in the time interval $[0, t]$.

When a point moves on the x-axis, we will denote its position function by $x(t)$. The initial position is $x(0)$; the velocity function is $v(t) = x'(t)$ and the acceleration is $a(t) = v'(t) = x''(t)$. If it moves along the y-axis, its position function will be written $y(t)$; its initial position is $y(0)$ and its velocity and acceleration are $v(t) = y'(t)$ and $a(t) = v'(t) = y''(t)$, respectively.

Suppose an object is in vertical free fall near the Earth's surface. Place a y-axis along the path of the fall with the origin at ground level. Let $y(t)$ be the position of the object at any time $t \geq 0$. The initial position of the object is $y(0)$ and its initial velocity is $v(0)$. See Figure 6.18. Recall from Chapter 4.2 that Galileo discovered that if the object has velocity $v(t)$ at time t and velocity $v(t+\Delta t)$ at a later time $t+\Delta$, then the ratio

$$\frac{v(t+\Delta t)-v(t)}{(t+\Delta t)-t}=\frac{v(t+\Delta t)-v(t)}{\Delta t}$$

is equal to the same constant C regardless of the time t of the first observation and the elapsed time Δt. Therefore, this ratio remains fixed at C when Δt is pushed to zero. It follows that for an object in free fall, the acceleration $a(t)$ at any time $t \geq 0$ is given by

$$a(t)=v'(t)=\lim_{\Delta t\to 0}\frac{v(t+\Delta t)-v(t)}{\Delta t}=C.$$

Since the acceleration is in the negative y-direction, the constant C is negative. Nowadays, the constant C is written as $-g$, and g is known to be equal to 32.17 feet/second^2, or 9.80 meters/second^2. The force F that produces the acceleration of $-g$ on the falling object is known as the *force of gravity*. By Newton's law, $F = -mg$, so that F depends on the mass m of the object. On the other hand, as Galileo observed, the constant g is the same regardless of the mass of the object.

Since the acceleration $a(t) = -g$ is a constant, the velocity and position functions of the objects are

$$v(t)=-gt+v(0) \quad\text{and}\quad y(t)=-\frac{g}{2}t^2+v(0)t+y(0).$$

This follows from Example 6.14 by replacing k by $-g$ and $p(t)$ by $y(t)$.

Example 6.15. Suppose Galileo had actually dropped a cannonball from the leaning tower of Pisa. Measure distance in feet and time in seconds. Assume that the initial velocity of the cannonball was $v(0) = 0$ and, since the tower is 177 feet high, that the initial position was $y(0) = 177$. Therefore,

$$y(t)=-\frac{g}{2}t^2+177.$$

Suppose the cannonball hit the ground at the instant t_1. So $y(t_1) = 0$ and $\frac{g}{2}t_1^2 = 177$. Since $g = 32$ feet/sec^2, it follows that $t_1^2 = \frac{354}{32}$, and $t_1 = \sqrt{\frac{354}{32}} = 3.3$ seconds.

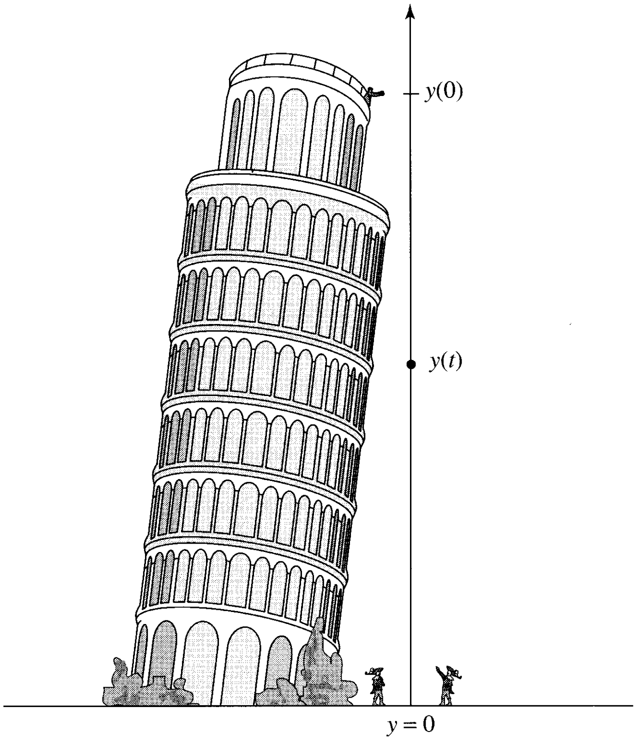


Figure 6.18

The velocity of the cannonball at impact was $v(t_1) = -gt_1 + v(0) = -32(3.3) = -106$ feet per second.

6.5 The Trajectory of a Projectile

We will now turn to the analysis of the trajectory of a projectile, which applies in principle to any object that is thrown and hence to projectiles fired by guns and cannons. The basic understanding is that the projectile is "left to its own devices" after being given an initial velocity at launch. In other words, it is not subject to any propelling forces thereafter.

Set up a coordinate system as shown in Figure 6.19. The projectile starts its motion at time $t = 0$, when its initial position is the point $(0, y_0)$. It has an initial velocity that is represented by the arrow in the

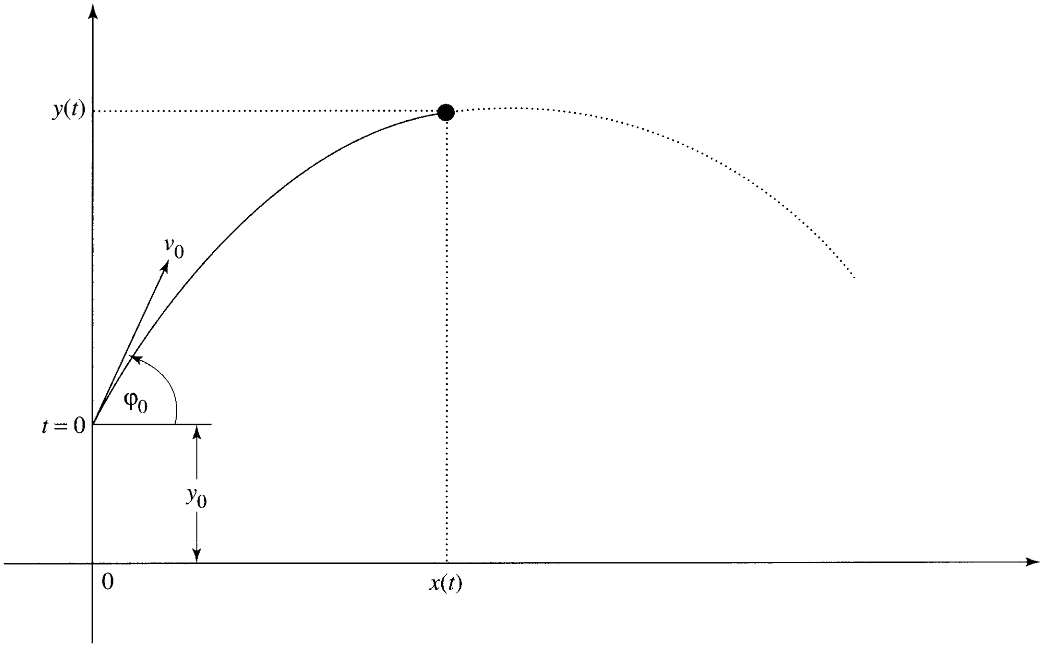


Figure 6.19

diagram. Such arrows are called *vectors*. The length of the vector is equal to the initial speed v_0 of the projectile. The direction of the vector with respect to the horizontal is known as the *angle of elevation* or *angle of departure*. It is given by the angle φ_0, where $0 \leq \varphi_0 \leq \frac{\pi}{2}$.

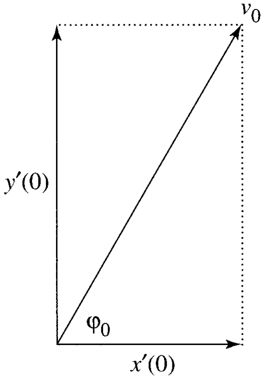


Figure 6.20

For any $t \geq 0$, the coordinates $x(t)$ and $y(t)$ are the perpendicular projections of the position of the projectile onto the x- and y-axes respectively. Note that $x(0) = 0$ and $y(0) = y_0$. By the analysis of Section 6.4, $x'(t)$ is the velocity of the projectile in the x-direction and $y'(t)$ is the velocity in the y-direction. So $x'(0)$ and $y'(0)$ are the initial velocities in the x- and y-directions. Together they determine the initial velocity vector by the *parallelogram law*. Place a horizontal vector of length $x'(0)$ and a vertical vector of length $y'(0)$ "tail to tail" as shown in Figure 6.20. The initial velocity vector is the diagonal of the parallelogram (in this case it is a rectangle) that results. The length of this diagonal is the initial speed v_0 of the projectile.

Refer to Figure 6.20 again and observe that $\cos\varphi_0 = \frac{x'(0)}{v_0}$ and $\sin\varphi_0 = \frac{y'(0)}{v_0}$. Therefore,

$$x'(0) = v_0 \cos\varphi_0 \quad \text{and} \quad y'(0) = v_0 \sin\varphi_0.$$

The only force on the projectile during its flight is gravity. It acts in the negative y-direction. In particular, the force in the x-direction is zero. Therefore, by Newton's law $F = ma$, the acceleration in the x-direction is also zero. Thus $x''(t) = 0$. Observe that Example 6.13 applies to the projection of the projectile onto the x-axis with $C = v_0 \cos \phi_0$. Thus

$$x'(t) = v_0 \cos \varphi_0 \quad \text{and} \quad x(t) = (v_0 \cos \varphi_0)t. \tag{6a}$$

Because gravity produces an acceleration of $-g$ in the y-direction, $y''(t) = -g$. Since $y'(0) = v_0 \sin \varphi_0$ and $y(0) = y_0$, we find by the discussion at the end of Section 6.4 that

$$\boxed{\begin{aligned} &y'(t) = -gt + v_0 \sin \varphi_0 \quad \text{and} \\ &y(t) = -\frac{g}{2}t^2 + (v_0 \sin \varphi_0)t + y_0 \end{aligned}} \tag{6b}$$

By the second equation of (6a), $t = \frac{x(t)}{v_0 \cos \varphi_0}$, so $t^2 = \frac{(x(t))^2}{v_0^2 \cos^2 \varphi_0}$. By a substitution into the expression for $y(t)$ in (6b), we get

$$\begin{aligned} y(t) &= \frac{-g}{2v_0^2 \cos^2 \varphi_0}(x(t))^2 + \frac{v_0 \sin \varphi_0}{v_0 \cos \varphi_0}x(t) + y_0 \\ &= \frac{-g}{2v_0^2 \cos^2 \varphi_0}(x(t))^2 + (\tan \varphi_0)x(t) + y_0. \end{aligned}$$

Observe, therefore, that the position $(x(t), y(t))$ of the projectile at any time t satisfies the following equation:

$$\boxed{y = \left(\frac{-g}{2v_0^2 \cos^2 \varphi_0}\right)x^2 + (\tan \varphi_0)x + y_0} \tag{6c}$$

By Section 4.3, this is the equation of a parabola. Therefore, the trajectory of the projectile is a parabola, indeed, a very specific parabola determined by the constants v_0, y_0, and φ_0.

What is the maximal height reached by the projectile? This maximum occurs at the instant at which the vertical component of the velocity is zero, i.e., at the instant t_1 at which $y'(t_1) = 0$. By the first equation of (6b), $0 = y'(t_1) = -gt_1 + v_0 \sin \varphi_0$, or $t_1 = \frac{v_0 \sin \varphi_0}{g}$. Inserting this t_1 into the second equation of (6b) shows that

$$\begin{aligned} y(t_1) &= -\frac{g}{2}t_1^2 + (v_0 \sin \varphi_0)t_1 + y_0 \\ &= -\frac{1}{2g}v_0^2 \sin^2 \varphi_0 + \frac{1}{g}v_0^2 \sin^2 \varphi_0 + y_0. \end{aligned}$$

Therefore the maximal height attained by the projectile is

$$\frac{1}{2g}v_0^2 \sin^2 \varphi_0 + y_0. \tag{6d}$$

For the rest of our discussion we will assume that the terrain is flat and the x-axis lies along the ground. At what time and how far downrange will the projectile hit the ground? Impact occurs precisely at the time t_{imp} for which $y(t_{\text{imp}}) = 0$. Refer to the second equation of (6b) and observe that the time of impact can be found by solving

$$-\frac{g}{2}(t_{\text{imp}})^2 + (v_0 \sin \varphi_0)t_{\text{imp}} + y_0 = 0$$

for t_{imp}. By the quadratic formula and the fact that $t_{\text{imp}} \geq 0$,

$$t_{\text{imp}} = \frac{v_0 \sin \varphi_0 + \sqrt{v_0^2 \sin^2 \varphi_0 + 2gy_0}}{g}. \tag{6e}$$

The *range* of the projectile is the horizontal distance from the initial position of the projectile to the point of impact. So the range is equal to the x-coordinate R of the projectile at the time of impact t_{imp}. It follows from (6a) that

$$\begin{aligned} R &= x(t_{\text{imp}}) = (v_0 \cos \varphi_0)t_{\text{imp}} \\ &= \frac{v_0^2}{g}(\sin \varphi_0)(\cos \varphi_0) \\ &\qquad + \frac{v_0}{g}\sqrt{(\cos^2 \varphi_0)v_0^2(\sin^2 \varphi_0) + 2gy_0(\cos^2 \varphi_0)}. \end{aligned}$$

The trigonometric formula $\sin 2\varphi_0 = 2 \sin \varphi_0 \cos \varphi_0$ (see Exercises 2C) now provides the following expression for the range:

$$R = \frac{v_0^2}{2g}\sin(2\varphi_0) + \frac{v_0}{g}\sqrt{\frac{v_0^2}{4}\sin^2(2\varphi_0) + 2gy_0\cos^2\varphi_0}. \tag{6f}$$

What angle φ_0 gives the maximal range, i.e., the largest possible R? Suppose $y_0 = 0$. A look at the expression in (6f) in this case shows that R is maximal when $\sin(2\varphi_0)$ is largest. Since $0 \le \varphi_0 \le \frac{\pi}{2}$, it follows that $0 \le 2\varphi_0 \le \pi$. The largest value of $\sin(2\varphi_0)$ in this interval is 1, and it occurs for $2\varphi_0 = \frac{\pi}{2}$, i.e., for $\varphi_0 = \frac{\pi}{4}$. So if $y_0 = 0$ the maximal range is

$$R_{\max} = \frac{v_0^2}{2g} + \frac{v_0}{g}\sqrt{\frac{v_0^2}{4}} = \frac{v_0^2}{2g} + \frac{v_0^2}{2g} = \frac{v_0^2}{g}. \tag{6g}$$

If $y_0 \neq 0$, then $\varphi_0 = \frac{\pi}{4}$ does not provide the greatest range R. However, the determination of the angle φ_0 that does so is beyond our current mathematical firepower.

The velocity of the projectile at any time t is determined by the velocities $x'(t)$ and $y'(t)$, by the parallelogram law (Figure 6.20), and the equations (6a) and (6b). Therefore, its speed at any time t is

$$\sqrt{(x'(t))^2 + (y'(t))^2}$$

$$= \sqrt{v_0^2\cos^2\varphi_0 + g^2t^2 - 2(gv_0\sin\varphi_0)t + v_0^2\sin^2\varphi_0}.$$

Since $v_0^2\cos^2\varphi_0 + v_0^2\sin^2\varphi_0 = v_0^2(\cos^2\varphi_0 + \sin^2\varphi_0) = v_0^2$, this is equal to

$$\sqrt{v_0^2 + g^2t^2 - 2g(v_0\sin\varphi_0)t}. \tag{6h}$$

The speed at impact is obtained by plugging the moment of impact t_{imp} into this formula. How can the angle of impact be determined? See Exercise 20.

A look at equations (6a)–(6h) shows that the initial conditions y_0, v_0, and φ_0 and the constant g determine everything about the motion. We conclude our discussion by illustrating the theory with a look at a numerical example.

Newton tosses an apple in the direction of his younger colleague, the astronomer Edmund Halley. When the apple leaves his hand, it is 3 feet above the ground, has an initial velocity of 25 feet per second, and has an angle of departure of 45°. The ground is level and Halley stands 18 feet away. Is he in a position to catch it? (See Figure 6.21.)

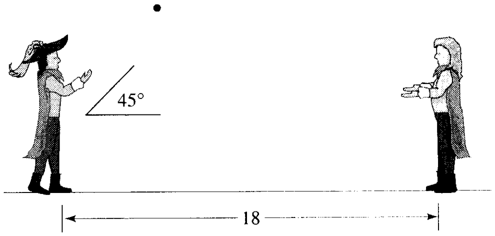


Figure 6.21

We assume that the toss occurs at time $t = 0$. The question that has to be asked is this: how high will the apple be when it reaches the spot where Halley is standing? This is a question about $y(t)$ or, more precisely, about the value of $y(t)$ at the time t when the apple reaches that spot. If we can determine the time when this occurs, then the use of the second formula in (6b) will provide the answer. But how do we get t? We know that $v_0 = 25$ feet per second and $\varphi_0 = 45°$. We also know that we are looking for t such that $x(t) = 18$. So by (6a),

$$18 = x(t) = (v_0\cos\varphi_0)t = (25\cos 45°)t = 25\frac{\sqrt{2}}{2}t,$$

and we get[3]

$$t = \frac{18}{25}\frac{2}{\sqrt{2}} = \frac{18}{25}\frac{2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{18}{25}\sqrt{2} \approx (0.72)(1.41)$$

$$= 1.02 \text{ seconds}.$$

Plugging $t = 1.02$, $g = 32$, $y_0 = 3$, and $\varphi_0 = 45°$ into $y(t) = -\frac{g}{2}t^2 + (v_0\sin\varphi_0)t + y_0$, gives us

$$y(1.02) \approx -16(1.02)^2 + (25\sin 45°)(1.02) + 3$$

$$\approx -16(1.04) + 25\frac{\sqrt{2}}{2}(1.02) + 3$$

$$\approx -16(1.04) + 25(0.71)(1.02) + 3$$

$$\approx -16.65 + 18.03 + 3 = 4.39 \text{ feet.}$$

So the apple will be about $4\frac{1}{2}$ feet above the ground when it gets to Halley. Alas, Halley ducks! How far behind him will the apple land? With what velocity will it hit the ground? Since the angle of departure is $\phi_0 = 45^\circ$, formula (6f) can be used to determine where the apple will hit the ground:

$$R = \frac{v_0^2}{2g} + \frac{v_0}{g}\sqrt{\frac{v_0^2}{4} + gy_0} = \frac{25^2}{64} + \frac{25}{32}\sqrt{\frac{25^2}{4} + (32)(3)}$$

$$\approx 9.77 + 0.78(\sqrt{156.25 + 96}) \approx 9.77 + 0.78(15.88)$$

$$\approx 9.77 + 12.41 \approx 22.17 \text{ feet.}$$

Since Halley stands 18 feet from Newton, the apple lands a little over 4 feet behind Halley. The time of impact is given by the formula

$$t_{\text{imp}} = \frac{v_0 \sin\varphi_0 + \sqrt{v_0^2 \sin^2\varphi_0 + 2gy_0}}{g}.$$

Inserting the constants, $v_0 = 25$, $y_0 = 3$, $\varphi_0 = 45^\circ$, and $g = 32$, we get

$$t_{\text{imp}} = \frac{1}{32}\left(25 \cdot \frac{\sqrt{2}}{2} + \sqrt{25^2 \cdot \frac{2}{4} + 2 \cdot 32 \cdot 3}\right)$$

$$\approx \frac{1}{32}\left(17.68 + \sqrt{312.5 + 192}\right)$$

$$\approx \frac{1}{32}(17.68 + 22.46) \approx \frac{40.14}{32} \approx 1.25 \text{ seconds.}$$

Inserting t_{imp} and the other constants into (6h) tells us that the speed of the apple at impact is

$$\sqrt{25^2 + 32^2(1.25)^2 - 2 \cdot 32 \cdot 25 \cdot \frac{\sqrt{2}}{2}(1.25)}$$

$$\approx \sqrt{625 + 1600 - 1414.21}$$

$$\approx 28.47 \text{ feet/second.}$$

What is the maximal height reached by the apple? This is given by (6d):

$$\frac{1}{2g}v_0^2 \sin^2\varphi_0 + y_0 = \frac{1}{64}25^2 \cdot \frac{1}{2} + 3 = \frac{625}{128} + 3$$

$$\approx 4.88 + 3 \approx 7.88 \text{ feet.}$$

Let's reflect over what Newton's calculus has achieved. It has isolated the gravitational aspect of the motion of a projectile and provided full understanding of it. In particular, equations (6a)–(6h) provide a complete description of the motion of a projectile in any situation where gravity is the only influencing factor. This is so, for instance, on the Moon (for a different constant g), where there is no atmosphere. Equations (6a)–(6h) also provide an accurate picture of what happens on Earth for any relatively heavy object moving at a low speed. In such cases gravity is often the dominant influence, and other factors are negligible. This is so for the apple just thrown by Newton and, as we will see in Section 9.3, for some of the important experiments of Galileo.

For a light object, or for an object moving at a high velocity, air resistance becomes a significant factor. Anyone who has ever stuck his hand out of the window of a car (moving at a speed of, say, 60 miles per hour = 88 feet per second) and felt the stiff resistance of the air has already experienced this. The equations (6a)–(6h) are based on the fundamental observation of Galileo that the increase in the velocity of a falling body over time is a constant. At high speeds, however, this observation is valid only in an idealized, air-resistance-free setting. The question that arises is this: Will these equations retain at least some of their value in a resistant medium, or will the impact of air resistance render them completely useless?

6.6 Applications to Ballistics?

Ballistics is the investigation of the motion of projectiles, primarily in the context of both light and heavy firearms. This aspect of the mod-

KIND OF ORDNANCE	Powder	Ball	Elevation	Range	REMARKS
	LBS		° '	YARDS	
6-PDR. FIELD GUN	1.25	Shot	0 00	318	
		"	1 00	674	
		"	2 00	867	
		"	3 00	1138	
		"	4 00	1256	
		"	5 00	1523	
	1.00	Sph. case shot	2 00	650	Time of flight 2 secs
			2 30	840	" " 3 "
		"	3 00	1050	" " 4 "
12-PDR. FIELD GUN	2.50	Shot	0 00	347	
		"	1 00	662	
		"	1 30	785	
		"	2 00	909	
		"	3 00	1269	
		"	4 00	1455	
		"	5 00	1663	
	1.50	Sph. case	1 00	670	Time 2 secs
		"	1 45	950	" 3 "
		"	2 30	1250	" 4 "

Figure 6.22

ern science of artillery has its origins in the middle of the 14th century, when gunpowder came into use in western Europe. At first, artillery was deployed mainly against fortifications. It was used extensively in the Thirty Years War of 1618–1648, and it has played an increasingly important role in warfare ever since.

In the 18th and 19th centuries, cannons were classified by the weight of the cannonball that they fired. For example, there were 6-pdr. (pounder), 9-pdr., 12-pdr., 24-pdr., and 32-pdr. cannons. Figures 6.22 and 6.23 contain data for 19th century American 6-pdr. and 12-pdr. field guns. These tables come from the appendix of *The Artillerist's Manual*,[4] originally published in 1860. In Figure 6.22, the column "Powder." refers to the quantity of powder used, the column "Ball." to the types of ammunition (made mostly of iron), and the column "Elevation" to the angle of departure.

Figure 6.23 (refer to the column Initial Velocity) lists the initial speed of the projectiles in feet per second for different types of ammunition and different quantities of powder used. These initial speeds, also called *muzzle velocities*, were determined by experiments with the ballistic pendulum at the Washington Arsenal. The operative principle

KIND OF ORDNANCE	PROJECTILE		Charge of Powder	Initial Velocity
	Kind	Weight		
		Lbs.	Lbs.	Feet
6-PDR. FIELD GUN......	Shot.............	6.15	1.25	1439
			1.50	1563
			2.00	1741
	Sph. case.........	5.50	1.00	1357
	Canister..........	6.80	1.00	1230
12-PDR. FIELD GUN.....	Shot.............	12.30	2.50	1486
			3.00	1597
			4.00	1826
	Sph. case.........	11.00	2.00	1262
	Canister.........	13.50	2.00	1392

Figure 6.23

behind a ballistic pendulum is as follows: A shell is fired at close range at a heavy pendulum, the maximal deviation of the pendulum is measured, and the muzzle velocity of the projectile is determined from the amount of the deviation and the weight of the projectile. (We will study the ballistic pendulum in Section 14.3.)

The data in Figures 6.22 and 6.23 are based on actual measurements and therefore take air resistance into account. The problem that now concerns us is this: How do these data compare to the predictions of the theory of Section 6.5, which ignores air resistance?

Let's consider, for instance, the 6-pdr. field gun standing on horizontal ground. Let's put in 1.25 pounds of powder, load a cannonball, set the angle of departure at 1°, and fire. From Figure 6.23 we see that the muzzle velocity is $v_0 = 1439$ feet per second. The muzzle of this gun in firing position is about $y_0 = 3.6$ feet above the ground. (This information comes from the appendix of *The Artillerist's Manual.*)

Let's first compute the range of the projectile using the theory of Section 6.5. We need to apply formula (6f),

$$R = \frac{v_0^2}{2g}\sin(2\varphi_0) + \frac{v_0}{g}\sqrt{\frac{v_0^2}{4}\sin^2(2\varphi_0) + 2gy_0\cos^2\varphi_0},$$

with $v_0 = 1439$, $y_0 = 3.6$, and $\varphi_0 = 1^\circ$. Plugging in these values, we obtain

$$\begin{aligned} R &= \frac{(1439)^2}{2 \cdot 32}\sin 2^\circ \\ &\quad + \frac{1439}{32}\sqrt{\frac{(1439)^2}{4}\sin^2 2^\circ + 2(32)(3.6)(\cos^2 1^\circ)} \\ &\approx 1129.17 + 44.97\sqrt{630.52 + 230.33} \\ &\approx 1129.17 + (44.97)(29.68) \\ &\approx 2448.57 \text{ feet.} \end{aligned}$$

The corresponding observed value from Figure 6.22 is 674 yards = 2022 feet. The prediction of the theory deviates from the observed value by about 20%.

Let's continue firing the 6-pdr. field gun. Put in 1.00 pound of powder, load a cannonball, set the angle of departure at 3°, and fire. From Figure 6.23, we see that the muzzle velocity is $v_0 = 1357$ feet per second.

Let's compute the time t_{imp} at which the cannonball will impact. By formula (6e),

$$t_{\text{imp}} = \frac{v_0 \sin\varphi_0 + \sqrt{v_0^2 \sin^2\varphi_0 + 2gy_0}}{g},$$

where $v_0 = 1357$, $y_0 = 3.6$, and $\varphi_0 = 3°$. Therefore,

$$\begin{aligned} t_{\text{imp}} &\approx \frac{(1357)(0.05) + \sqrt{((1357)(0.05))^2 + 2(32)(3.6)}}{32} \\ &\approx \frac{67.9 + \sqrt{5043.83 + 230.40}}{32} \\ &\approx \frac{71.02 + 72.62}{32} = \frac{143.64}{32} \\ &\approx 4.49 \text{ seconds.} \end{aligned}$$

Notice that this is in reasonable agreement with the observed 4 seconds that Figure 6.22 provides.

Consider formula (6f) next. Taking

$$R = \frac{v_0^2}{2g}\sin(2\varphi_0) + \frac{v_0}{g}\sqrt{\frac{v_0^2}{4}\sin^2(2\varphi_0) + 2gy_0\cos^2\varphi_0}$$

with $v_0 = 1357$, $y_0 = 3.6$, and $\varphi_0 = 3°$, we get

$$\begin{aligned} R &= \frac{(1357)^2}{2 \cdot 32}\sin 6° \\ &\quad + \frac{1357}{32}\sqrt{\frac{(1357)^2}{4}\sin^2 6° + 2(32)(3.6)(\cos^2 3°)} \\ &\approx 3007.56 + 42.41\sqrt{5030.01 + 229.77} \\ &\approx 6083.04 \text{ feet.} \end{aligned}$$

The observed range, on the other hand, is 1050 yards = 3150 feet. This time the gap between theory and observation is big.

While air resistance has little effect when velocities are small, the lesson is that its effect at larger velocities can be substantial. In the context of artillery, the formulas of Section 6.5 are of little or no value. Indeed, at high velocities and short ranges, the effect of air resistance is much greater than that of gravity. The shape or, more precisely, the *aerodynamics* of the projectile play a crucial role. By relating this factor to the velocity, it is possible to take it into account by modifying the basic air-resistance-free theory of the trajectory of a projectile. In simple situations, such as free fall, this is relatively simple (see Section 13.6). In general, however, the mathematical relationship between the aerodynamics and the velocity depends on the velocity. It is hardly surprising that mathematical and computer models of trajectories that take air resistance into account are extremely complicated. See Exercises 6F for yet another complicating factor.

6.7 Postscript

Hooke and Newton, in spite of disagreements and quarrels, continued to correspond about matters of science and mathematics. In one letter, Hooke suggested that the planets are kept in orbit by a central force that decreases with increasing distance; indeed he suggested that the central force decreases with the square of the distance. It seems clear, however, that Hooke had no mathematical proof. This correspondence revived Newton's interest in these fundamental questions. Edmund Halley became interested as well. Halley visited Cambridge in 1684 and asked Newton specifically what the shape of the orbit of a planet would be under the assumption that it is attracted by a force centered at the Sun and inversely proportional to the square of its distance from the Sun. Newton immediately answered that the orbit would be an ellipse, indicating that he had proved this long ago. Later he supplied Halley with two proofs. Halley realized the importance of this work at once and, with the support of the Royal Society, decided to persuade

Newton to publish a treatise setting out his discoveries in detail. In spite of his reluctance to publish, Newton finally agreed, and started this task in 1685. In spite of a number of ongoing problems—including disputes about the priority of the discovery of the inverse square law (principally with Hooke) and the financing of the printing (all expenses were met by Halley)—the book appeared in the summer of 1687. Entitled *Philosophiae Naturalis Principia Mathematica*, and known simply as the *Principia*, it is regarded by most scientists as the greatest book of science ever written. We'll look at it in the next chapter.

After the publication of the *Principia*, Newton's scientific output declined. In 1689 he was elected as Cambridge University's Member of Parliament.[5] In 1696 he moved to London and assumed the post of Warden of the Mint. He took his duties there very seriously and oversaw the replacement of the coinage of England. In 1699 he became the head, or Master, of the Mint, a position that he retained until his death in 1727. In 1703 he was elected president of the Royal Society, and in 1704 his work *Optiks* appeared. By that time he was famous throughout the scientific world. In 1705, Queen Anne conferred knighthood on Newton.

Disputes continued, especially with Leibniz. The point of contention was as to whether Leibniz had invented his calculus of differentials independently or whether he had "borrowed" from Newton. The battle, fought out in the scientific literature, was contentious and bitter. It was waged not only by the two scientists themselves, but also by their supporters. Today's scientific community is in unanimous agreement that the two men had invented their versions of the calculus independently. A comparison of the two theories brings the conclusion that while Leibniz had the better theory (from the point of view of notation and basic concepts), Newton's was much more powerful computationally.

The reviews on Newton the man are mixed. A look at his personal library reveals substantial interests in religion as well as alchemy. A look at his character finds him to be noble and sensitive, but also petty, suspicious, and withdrawn. He was a man who was admired and revered, but he aroused little affection.

Exercises
6A. Derivatives

1. Compute the derivatives of the following functions:

- **i.** $f(x) = 3\sqrt{x}$
- **ii.** $g(x) = \frac{4}{x^5}$
- **iii.** $h(x) = -\frac{2}{\sqrt[3]{x}}$
- **iv.** $f(x) = \frac{5}{x^{100}} - 4x^{-\frac{1}{3}}$
- **v.** $g(x) = -2x^{\frac{1}{3}} + 3x^5 - 6$
- **vi.** $y = x^{-\frac{2}{7}} + 30x^4 - \frac{1}{4}x^{\frac{5}{3}}$

6B. Antiderivatives and Definite Integrals

2. Determine antiderivatives of the following functions:

- **i.** $f(x) = 2x^3$
- **ii.** $f(x) = 5x^{\frac{1}{3}}$
- **iii.** $f(x) = 3x^5 + \frac{1}{4}x^{\frac{2}{7}}$
- **iv.** $f(x) = 6x^4 - \frac{3}{8}x^{\frac{5}{3}}$

3. Compute the areas under the graphs of the following functions:

- **i.** $f(x) = 5x^2$ between 0 and 4; between 4 and 6.
- **ii.** $f(x) = 5x^2$ between 0 and 6.
- **iii.** $f(x) = 3\sqrt{x}$ between 0 and 5.
- **iv.** $f(x) = 4x^3 + 2x^{\frac{1}{3}}$ between 2 and 4.

4. Make use of the approximation

$$\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \dots$$

for $|x| < 1$ to compute the definite integrals

$$\int_0^{\frac{3}{4}} \frac{1}{1+x}\,dx \quad \text{and} \quad \int_0^{\frac{3}{4}} \frac{1}{1+x^2}\,dx$$

with an accuracy of four decimal places.

5. The approximation

$$\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \dots$$

for $|x| < 1$, can be used to compute related definite integrals. For example, multiplication by $x^{\frac{1}{2}}$ shows that

$$\frac{x^{\frac{1}{2}}}{1+x} \approx x^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}} - x^{\frac{7}{2}} + x^{\frac{9}{2}} - \dots .$$

Use this approximation to show that

$$\int_0^{\frac{1}{2}} \frac{x^{\frac{1}{2}}}{1+x}\,dx \approx 0.184.$$

6. Make use of the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

to evaluate the definite integrals

$$\int_0^{\pi} \sin x\,dx \quad \text{and} \quad \int_0^1 \sin\sqrt{x}\,dx$$

with four decimal accuracy.

7. Try to make use of the approximation

$$\begin{aligned}\sqrt{1+x} &= (1+x)^{\frac{1}{2}}\\ &\approx 1 + \binom{\frac{1}{2}}{1}x + \binom{\frac{1}{2}}{2}x^2 + \binom{\frac{1}{2}}{3}x^3\\ &\quad + \binom{\frac{1}{2}}{4}x^4 + \dots\end{aligned}$$

to compute the area under the graph of $f(x) = \sqrt{1+x}$ between 0 and 5. Why is there a problem?

8. Use the formula $L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$ together with a power series approximation to compute the length of the graph of $f(x) = \frac{1}{3}x^3$ between $(-1, -\frac{1}{3})$ and $(1, \frac{1}{3})$ with three decimal accuracy.

6C. Moving Points

Exercises 9–12 all involve the following situation: A particle is moving on a coordinatized line. Its position function $p(t)$ and the starting time are given. In each case, determine the particle's velocity $v(t)$ and acceleration $a(t)$ as functions of t, and give a description of the motion of the particle (including comments about the initial position, velocity, acceleration, and force; when and where the particle stops, etc.). Sketch the motion in each case (as in Figure 6.15 of the text).

9. $p(t) = 2t - 5$; start at $t = 0$.

10. $p(t) = 2t^2 + 2t + 12$; start at $t = -10$.

11. $p(t) = t^3 - 4t^2 - 21t$; start at $t = -6$.

12. $p(t) = \frac{3}{t} = 3t^{-1}$; start at $t = 1$.

13. A point is moving on a coordinatized line. It starts at the origin at time $t = 0$ with an initial velocity of zero. Its acceleration is given by $a(t) = 6t - 12$. (a) Determine its velocity function $v(t)$ and its position function $p(t)$. (b) Draw a careful diagram of the motion over the time interval $[0, 7]$. Indicate the direction of the motion and point out where and at what time(s) the point stops.

14. The acceleration $a(t)$ of a moving point is given by $a(t) = 2t - 6$ for $t \geq 0$. The initial velocity is $v(0) = 5$, and its initial position is $p(0) = 6$. Determine the velocity and position functions $v(t)$ and $p(t)$ for $t \geq 0$, and describe the motion of the point.

6D. Projectiles

15. Isaac Newton throws an apple in the direction of his nemesis, the scientist Robert Hooke, who stands on level ground 35 feet away. Newton throws the apple with an initial velocity of 40 feet per second and an angle of elevation of 20°. The apple is 5 feet above the ground at the moment of release. What is the maximal height above the ground reached by the apple? Will the apple hit Hooke? If so, with what speed?

16. Newton's pet parakeet escapes and flies over his garden straight towards his house. It has a speed of 20 feet per second and flies at an angle of 30° with the horizontal when it suddenly releases, at a height of 50 feet… a dropping. How long after the release will the dropping reach its maximal height? What is the maximal height? Will it splatter against the white stucco wall of Newton's house, which is 20 feet high and 30 feet away (at the time of the release)? If so, with what velocity?

17. At the opening ceremonies of the 1992 Olympic games in Barcelona, an archer lit the Olympic flame by shooting a burning arrow into a circular pool of flammable liquid. Suppose the circular pool was affixed to a structure that put the center of the surface of the liquid 55 meters downrange from the archer and 25 meters above the ground. Now suppose the archer shot the arrow from 1.5 meters above ground level at an angle of 70° with the horizontal and that the arrow struck the center of the pool on its descent. With what initial speed did the arrow leave the bow? [Hint: Ignore air resistance and use Formula (6c) of Section 6.5.]

18. Suppose that the Olympic archer in Exercise 17 shot the arrow from 6 feet above ground level and that the center of the circular pool was 240 feet downrange from the archer and 62 feet above the ground. If the initial speed of the arrow was 120 feet per second, determine the angle of elevation with which the marksman would have had to shoot the arrow in order to hit the center of the surface of the liquid. [Hint: Use the identity $\sec^2\varphi = 1 + \tan^2\varphi$.]

19. Suppose that a basketball player's jump shot is most accurate when he releases the ball from 8 feet above

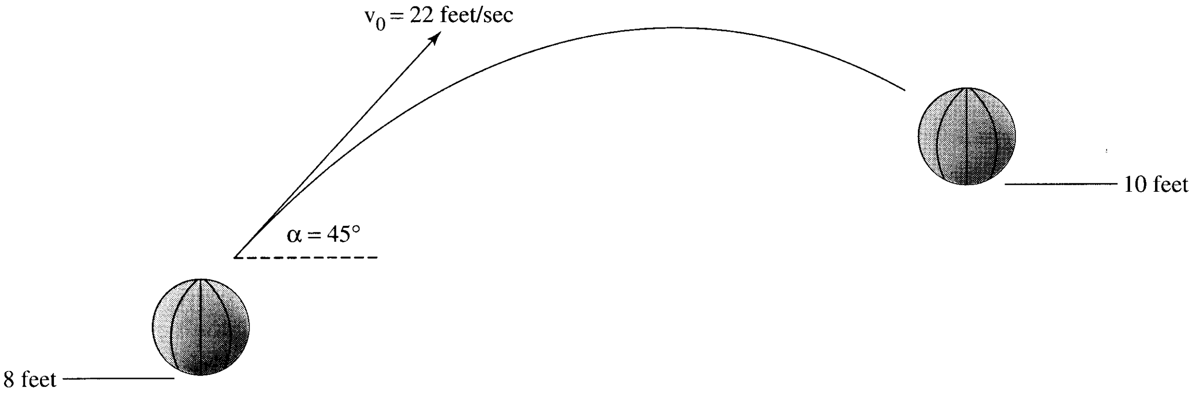


Figure 6.24

the floor at an angle of 45° and a velocity of 22 feet per second. See Figure 6.24. How far from the basket should he be taking his shots?

20. Using the formulas and notation developed for the motion of a projectile, derive a formula for the angle of impact of a projectile. Assume that the ground is horizontal.

6E. Ballistics

21. Load a 6-pdr. field gun with 1.25 pounds of powder. Put in a cannonball and fire the gun with an angle of departure of 0°, i.e., point blank. By Figure 6.23, the muzzle velocity of the cannonball is 1439 feet per second. Use the formulas developed in Section 6.5 to compute the range. Compare your result with the observation recorded in Figure 6.22. Why is your answer inconsistent with the data of Figures 6.22 and 6.23? What is the problem?

22. Now load a 6-pdr. field gun with sperical case shot and 1.00 pounds of powder. Fire with an angle of departure of 2°30′. By Figure 6.23, the muzzle velocity of the projectile is 1357 feet per second. Using the theory of Section 6.5 determine how many seconds after the firing the projectile will impact, and compute the range. Compare your figures with the observed values in Figure 6.22.

23. Fire a 12-pdr. field gun with a cannonball using 2.50 pounds of powder. By Figure 6.23, the muzzle velocity is 1486 feet per second. Fire three successive times with angles of departure of 0°, 1°, and 5°. Compare the ranges predicted by theory with those given in Figuree 6.22.

6F. Connections with Probability Theory

Another difficulty encountered in ballistics is illustrated by data in the appendix of *The Artillerist's Manual.* A certain type of rifle-musket was testfired a total of fifteen times on July 11 and again on July 23 of 1856. The initial velocities, measured with a ballistic pendulum, were found to be 809, 901, 878, 884, 850, 873, 927, 943, 822, 827, 870, 900, 897, 920, and 914 feet per second. Therefore, each firing of the musket produced a different initial velocity, even though the conditions (the weight of the powder, the type of ball, etc.) were kept the same! The number 884 has the property that there are seven velocities below it (809, 878, 850, 873, 822, 827, and 870) and seven above it (914, 901, 927, 943, 900, 897, 920, and 914). Thus 884 is the so-called *median* of the numbers on the list.

The probability that a certain event, often called a *favorable outcome,* occurs is a number between 0 and 1: It is the ratio of the number of all possible favorable outcomes to the number of all possible outcomes. To say that the probability is 0 means that the event will definitely not occur; to say that it is 1 means that it is certain to occur. In reference to the data just given, observe that 9 out of the 15 firings produced a velocity between 860 and 920 feet per second, so the probability that the initial velocity of the ball lies in this range is $\frac{9}{15} = 0.6$.

The data can be organized as in Figure 6.25. Observe that the probability just computed is equal to the sum of the areas of the rectangles between 860 and 920 over the total area of all the rectangles.

If the probability of 0.6 is to be reliable, many more tests than these 15, say several thousand, would be neces-

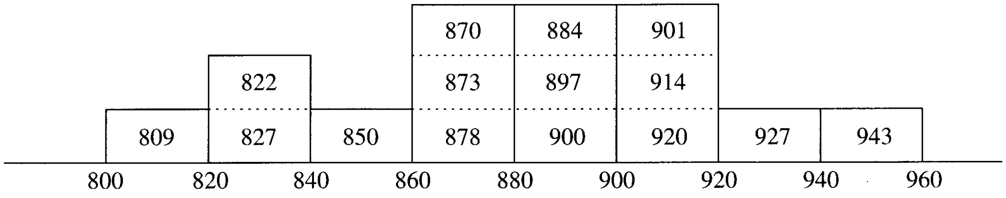


Figure 6.25

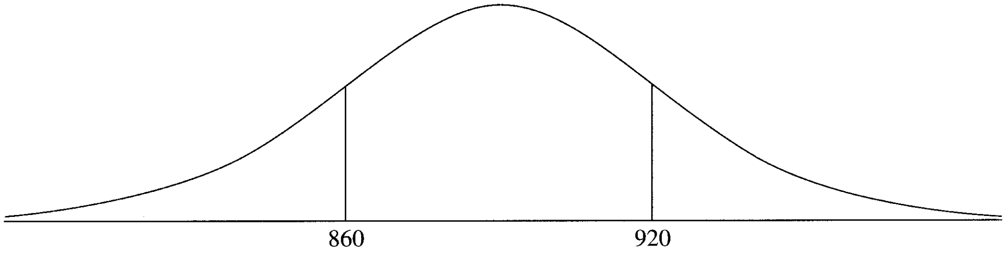


Figure 6.26

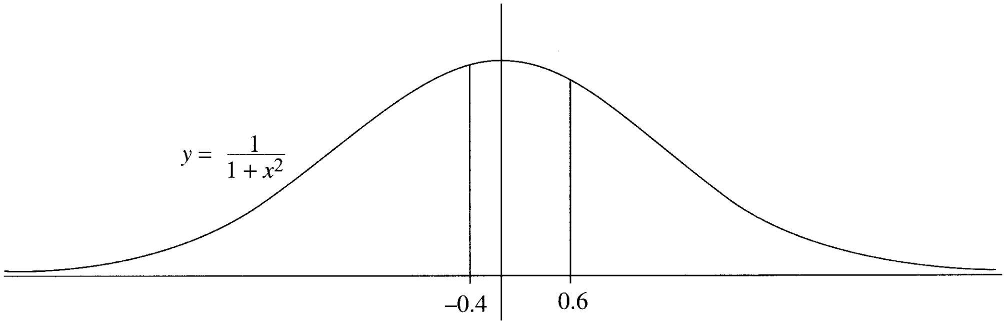


Figure 6.27

sary. If this were done, the array of Figure 6.25 would look something like the "bell-shaped" curve of Figure 6.26. The probability that a given firing of the rifle generates an initial velocity between 860 and 920 feet per second is the ratio of the area under the curve between 860 and 920 to the total area under the curve.

This discussion illustrates the fact that any accurate and useful mathematical theory of ballistics is necessarily complicated and necessarily involves probability theory. Such a theory will be able to assert, for example, that a projectile fired from a certain gun under certain conditions will land within 5 feet from a certain point with a probability of 0.9, or 0.95, or 0.99. Probability theory is beyond the purpose and aim of a basic text such as this.

24. In reference to the test firings just discussed, assume that after one thousand firings, the results can be modeled as follows. The curve of Figure 6.27 is the graph of the function $y = \frac{1}{1+x^2}$, where the value $x = 0$ corresponds to the median velocity of 884 feet per second, and every 0.1 unit on the x-axis represents 1 foot per second. So the region under the graph for $-0.4 \le x \le 0$ corresponds to all the firings with velocities v satisfying $880 \le v \le 884$, and the region under the graph for $0 \le x \le 0.6$ corresponds to all the firings with velocities v satisfying $884 \leq v \leq 900$. Assuming that the area under the entire graph is π, compute the probability that a particular test yields a velocity v such that $880 \leq v \leq 884$. Then compute the probabilities that $884 \leq v \leq 900$ and $880 \leq v \leq 900$. Is there a reasonable correlation between your findings and the data in Figure 6.25? [Hint: Compute the required definite integrals by using Newton's approximation method.]

Notes

[1] All references to Newton's work can be found in D.T. Whiteside, *The Mathematical Papers of Isaac Newton, Volumes I–VII* (Cambridge University Press, Cambridge, 1969).

[2] Incidentally, your calculator relies on power series in computations of logarithms and trigonometric quantities.

[3] In all calculator computations in this book, the answers will be rounded off only after the final step. This will minimize roundoff errors.

[4] John Gibbon, *The Artillerist's Manual*, Greenwood Press, Westport CT, 1971. Originally published by D. Van Nostrand Co., New York, and Trübner & Co., London, 1860.

[5] Isaac Newton spoke in the House of Commons only once. As the great man arose a hush descended on the House in anticipation of what he would have to say. Newton observed that there was a window causing a draft and asked if it could be closed.

7

The *Principia*

It is one of Newton's major achievements (although not his achievement alone) that he discovered the fundamental concepts that underlie motion. The *Principia* begins with these concepts. The *mass* of an object, or body (which is the word Newton uses), is defined to be density times volume. *Inertia* is the "power of resisting, by which every body as much as in it lies, perseveres in its state, either of rest, or of uniform motion, in a right [straight] line."[1] A *force* is an "action exerted upon a body, in order to change its state, either at rest, or of uniform motion, in a right line." The basic laws of motion follow next:

> **LAW I.** That every body perseveres in its state of resting, or of moving uniformly in a right line, as far as it is not compelled to change that state by an external force impressed upon it.
>
> **LAW II.** That the change of motion is proportional to the moving force impressed; and is produced in the direction of the right line, in which that force was impressed.
>
> **LAW III.** That reaction is always contrary and equal to action; or, that the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Let's elaborate on what Newton is saying. We will take the meaning of force from everyday experience as any action of pulling or pushing. A force has a magnitude that can be measured. For example, an amount of push or pull can be measured by the amount of the displacement it produces on some standardized steel spring. A force also has a direction. Its magnitude and direction together determine the force.

Let a force act on a body. Newton's second law says that the magnitude F of the force is directly proportional to the acceleration a that it produces in the motion of the body, and the mass m is the constant of proportionality. In other words, force = mass × acceleration, or $F = ma$. Measure F and measure a, and the mass m of the body is determined by $m = \frac{F}{a}$. Since $a = \frac{F}{m}$, an increase (or decrease) in F produces a corresponding increase (or decrease) in a. The second law also says that the direction of the acceleration is the same as that of the force. Galileo discovered (see Sections 4.2 and 6.4) that the acceleration produced on a body by the force of gravity F is equal to the constant $-g$, regardless of the mass m of the body. In this case, Newton's equation is $F = -mg$. The direction of the force is downward (towards the center of the Earth) and its magnitude mg is called the *weight* of the body. Since g is known (32 feet per second^2, or 9.8 meters per second^2), the mass m of the body can be determined from its weight mg.

Notice that Newton's first law is a direct consequence of his second law: If the magnitude of a force is zero, then the acceleration it produces in the motion of the body must be zero, and hence its velocity is constant. Observe that this is Galileo's law of inertia.

The third law is the assertion that for every force there is always an equal and opposite force. If you lean against a wall, the wall will push back on you with an equal and opposite force. If this force did not exist you would fall. So forces always occur in pairs. It is important to note that these two forces never act on the same body. Your leaning against a wall is a force on the wall. The wall pushing back is a force on you.

After a discussion of the "method of prime and ultimate ratios," in other words some elements of differential calculus, the *Principia* turns to the study of centripetal force. A *centripetal force* is one "by which bodies are drawn, impelled, or any way tend towards a point, as to a centre." So a centripetal force is one that always acts in the direction of a single fixed point. This fixed point is called the *center of force*. What is going on is most easily illustrated as follows. Take a string and tie an object P to one end. Hold the other end of the string and twirl the object as shown in Figure 7.1. Keep this end fixed at S and note that P will move in a circle. The force with which the string pulls on P in the process always acts in the direction of S. It is therefore a *centripetal force*. Newton's third law is illustrated also. As the object is twirled, there is a force pulling on the hand at S in the direction of P. It

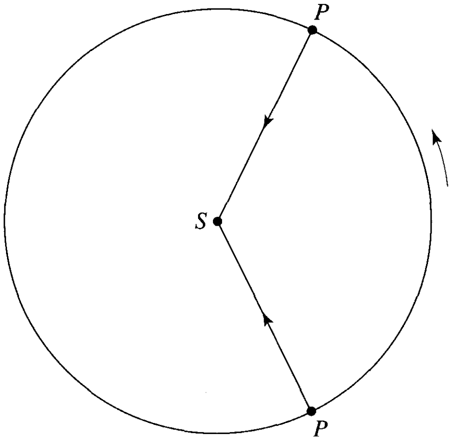


Figure 7.1

acts in the opposite direction but is equal in magnitude to the force pulling on P.

Everything described thus far is in Book I of the *Principia*. Book I goes on to develop a theory of bodies moving under the action of a centripetal force. This theory is completely abstract. It applies to any centripetal force acting on an object, be it the force exerted by a string on a twirling object, or the gravitational force of the Sun on a planet in its orbit. Book II of the *Principia* deals with motion in a resisting medium. But the high point of the *Principia* comes in Book III, the *System of the World*. It applies the theory of Book I to the analysis of the orbits of the planets.

It is the goal of this chapter to provide a detailed look at some of the central elements of Book I and then to illustrate the incredible insights that they provide into the workings of the solar system. The *Principia's* arguments are subtle, difficult, indeed often terse and opaque. It is, in short, quite a challenge to understand it. One reason for this is that Newton often used delicate Greek geometry known to the trained mathematicians of his time but not today's reader. In order to give a glimpse at Newton's explanations and also to facilitate the understanding of what he does, this chapter takes a two-fold approach. Newton's original arguments (from Thorp's 1777 translation) will appear in italic (*slanted*) print, and explanations are added in regular print. Figures labeled "Newton" are Newton's original diagrams (or parts thereof). Others have been added to explain and clarify.

7.1 Equal Areas in Equal Times

Consider an object or body in motion. Assume that it is acted upon by only one force, a continuously acting centripetal force. The magnitude of the force can vary. With later application to the motion of the planets around the Sun in mind, Newton denotes the body by P and the center of force by S. For a similar reason, we will refer to the path traced out by P as the *orbit* of P.

Newton begins his analysis by showing that the motion of the body obeys Kepler's second law. In other words, the single assumption that P moves under the action of a centripetal force with center S is enough to guarantee that the segment PS sweeps out equal areas in equal times. This single assumption also implies that the orbit of P lies in a plane that also contains S. See Figure 7.2.

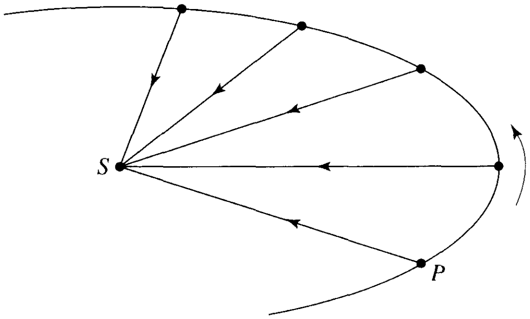


Figure 7.2

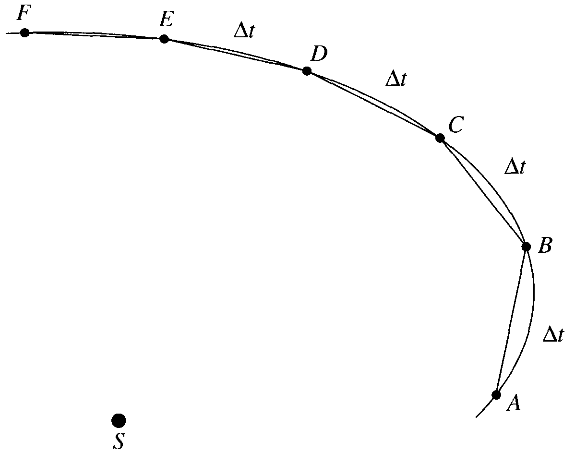


Figure 7.3

Newton takes a fixed interval of time Δt. He considers the position of the body at some point A, then Δt later at B, then another Δt later at C, another Δt later at D, etc. Newton regards the force to be acting only at the points A, B, C, D, etc., as a sequence of "impulses." Each impulse deflects the body in the same way that a billiard ball is deflected by the cushion of the table. In particular, he thinks of the actual curved path of the body as being approximated by a sequence of line segments. See Figure 7.3. If Δt is small, this sequential action of the force and the resulting approximation of the path of the body is a simulation of what is going on. The *Principia* now turns to the verification of Kepler's second law.

PROPOSITION I. THEOREM I. *That the areas, which revolving bodies describe by radii drawn to an immovable centre of force, do both lie in the same immovable planes, and are proportional to the times in which they are described.*

For suppose the time to be divided into equal parts (this is the Δt in the discussion above), *and in the first part of that time, let the body by its innate force describe the right line AB.* (Figure 7.4) *In the second part of that time, the same would, (by Law I.) if not hinder'd, proceed directly to c, along the line Bc equal to AB.* Since the force is (for the moment) considered not to be acting at B, there is no force acting on the body during the entire motion between A and c. So the velocity remains constant between A and c. The elapsed time from A to B is Δt and that from B to c is Δt also. It follows that $AB = Bc$. *So that by the radii AS, BS, cS drawn to the center, the equal areas ASB, BSc, would be described.* The triangles ΔSAB and ΔSBc have equal bases $AB = Bc$. They also, see Figure 7.5, have the same height h. So Area ΔSAB = Area ΔSBc.

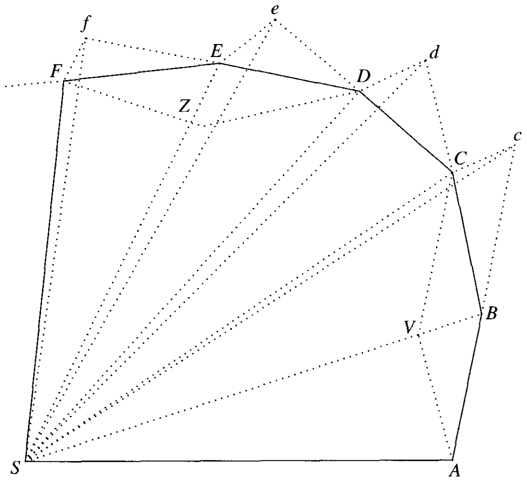


Figure 7.4 (*Newton*)

But when the body is arrived at B, suppose that a centripetal force acts at once with a great impulse, and turning aside the body from the right line Bc, compells it afterwards to continue its motion along the right line BC. In other words, the motion of the body from B to C is regarded as occurring in two components: first, the motion from B to c determined by the velocity of the body at B, and then the motion from c to C, which is determined by the action of the impulse at B along BS. *Draw cC parallel to BS meeting BC in C; and at the end of the second part of the time, the body (by Cor. I. of the laws) will be found in C, in the same plane with the triangle ASB.* Since the impulse at B lies in the plane of

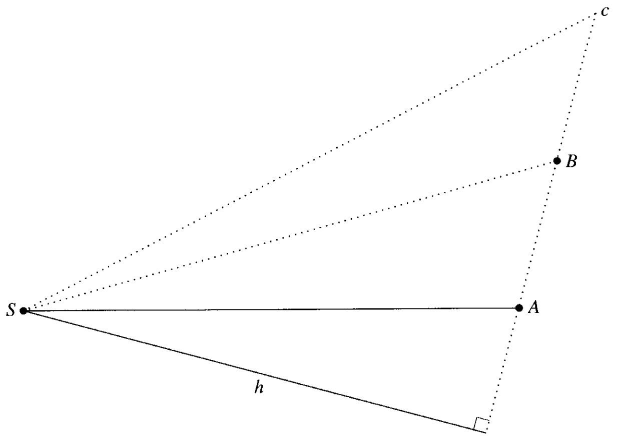


Figure 7.5

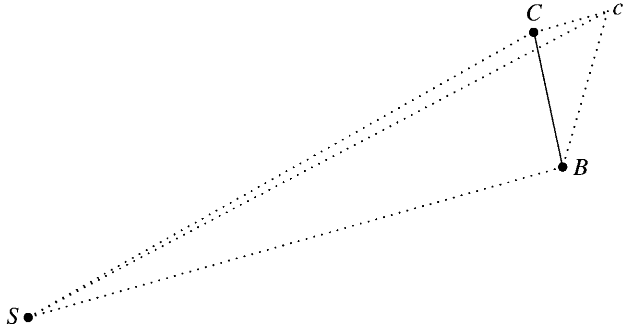


Figure 7.6 (Newton)

the triangle ΔSAB, there is no component of this force that could move the body outside this plane. *Joyn SC, and, because SB and Cc are parallel, the triangle SBC will be equal to the triangle SBc and therefore also to the triangle SAB.* The two triangles ΔSBC and ΔSBc have a common base SB. See Figure 7.6. Since Cc is parallel to SB, these triangles have the same height. Therefore, Area ΔSBc = Area ΔSBC. Since it was already shown that Area ΔSBc = Area ΔSAB, it follows that Area ΔSAB = Area SBC, as asserted by Newton.

By the like argument, if the centripetal force acts successively in C, D, E, &c. and makes the body in each single particle of time, to describe right lines CD, DE, EF, &c. they will all lye in the same plane; and the triangle SCD will be equal to the triangle SBC, and SDE to SCD, and SEF to SDE. And therefore in equal times, equal areas are describ'd in one immoveable plane: and, by composition, any sums SADS, SAFS, of those areas, are one to the other, as the times in which they are describ'd. Now let the number of those triangles be augmented, and their breadth diminished in infinitum; and (by Cor. 4. Lemma 3) their ultimate perimeter ADF will be a curve line: and therefore the centripetal force, by which the body is perpetually drawn back from the tangent of this curve, will act continually; and any describ'd areas SADS, SAFS, which are always proportional to the times of description, will, in this case also, be proportional to those times.

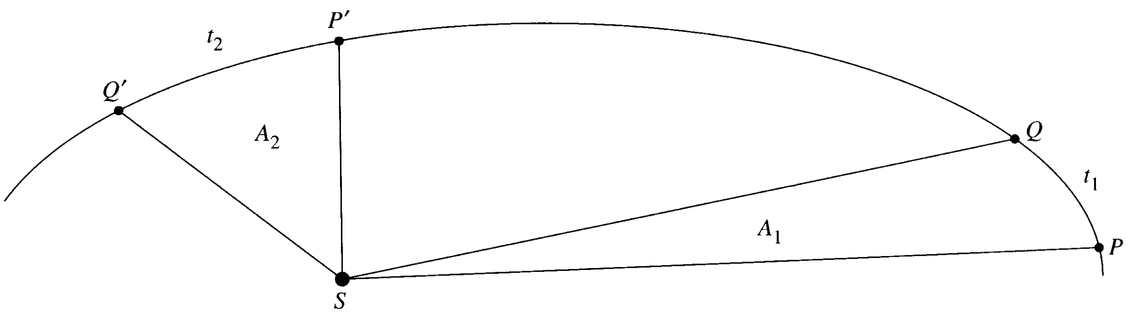


Figure 7.7

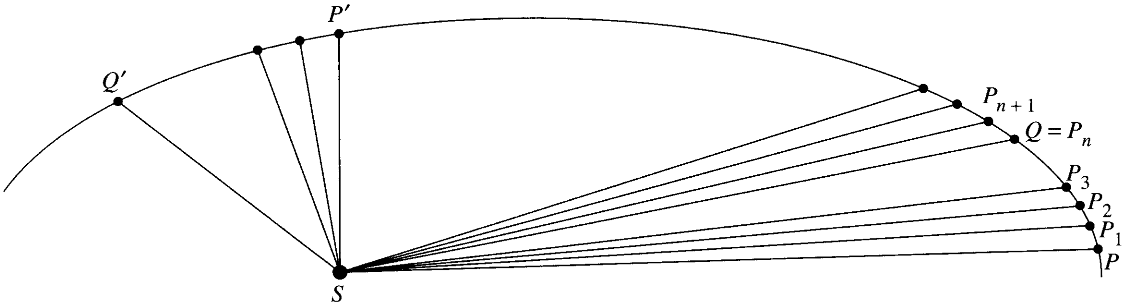


Figure 7.8

This argument, which completes Newton's proof of Kepler's law, requires explanation. Suppose that t_1 and t_2 are any two time intervals. Suppose that the body moves from P to Q during t_1 and from P' to Q' during t_2. Let the areas of the regions SPQ and $SP'Q'$ be A_1 and A_2 respectively. See Figure 7.7. Newton must demonstrate that $t_1 = t_2$ implies $A_1 = A_2$. What he does is this. Choose a small time interval Δt and partition t_1 into, say, n (possibly a huge number) of equal intervals, each of length Δt. So, $t_1 = n(\Delta t)$. Starting from P insert into the diagram the positions P_1, P_2, $P_3, \ldots$ of the body in its orbit at times Δt, $2\Delta t, 3\Delta t, \ldots$ from when it was at P, and continue this process until Q has been reached (Figure 7.8). Since n steps are needed to get from P to Q, Newton's construction inscribes the n triangles SPP_1, SP_1P_2, $SP_2P_3, \ldots$ into the region SPQ. By the step already verified, the triangles SPP_1 and SP_1P_2 have the same area. By the same step, the areas of the triangles SP_1P_2 and SP_2P_3 are the same. Continuing in this way, all the n triangles inscribed in the region SPQ have the same area, say ΔA. Since Δt is small, P and P_1, P_1 and P_2, P_2 and P_3, etc., are close together, and the n inscribed triangles together essentially fill out the area of the region SPQ. The area of this region is, therefore, $A_1 = n(\Delta A)$. Continue to insert points into the orbit at time intervals Δt, from Q to P', and finally from P' to Q'. Since $t_2 = t_1 = n(\Delta t)$, n steps are also required to get from P' to Q'. The triangles inscribed into the sector $SP'Q'$ in the process continue to have area ΔA. So the region $SP'Q'$ is also filled out by n triangles each with area ΔA. It follows that the area A_2 of this region is also equal to $n(\Delta A)$. So Newton has established Kepler's second law.

Strictly speaking, $A_1 = n(\Delta A)$ is not an equality, but only an approximation. This is because the triangles will not fill out the region SPQ "on the nose." The same is true for $SP'Q'$ and $A_2 = n(\Delta A)$. However, if the above thought experiment is repeated, each time for smaller and smaller Δt (and therefore larger and larger n) the approximations $A_1 = n(\Delta A)$ and $A_2 = n(\Delta A)$ get better and better: a rapid-fire sequence of impulses will approximate the continuously acting force, a myriad of connected line segments will approximate the curved trajectory, and the approximation $n(\Delta A)$ will (for all computational purposes) produce the equality $A_1 = nA = A_2$.

Let's formulate Newton's conclusion in another way as follows: Let the motion of a body P be produced by the action of a centripetal force acting towards the fixed point S (as in Figure 7.2). Let t be any time interval, and let A_t be the area of the sector that the segment SP sweeps out during the time t. Then the ratio $\frac{A_t}{t}$ is the same, regardless of what t is taken. To see this (we have in fact already seen this in Section 4.1 in the context of Kepler's second law), let t_1 and t_2 be any two time intervals and let A_1 and A_2 be the respective areas that are swept out. Put $\frac{t_1}{t_2} = c$. So $t_1 = ct_2$. Since A_2 is swept out in t_2, cA_2 is swept out in

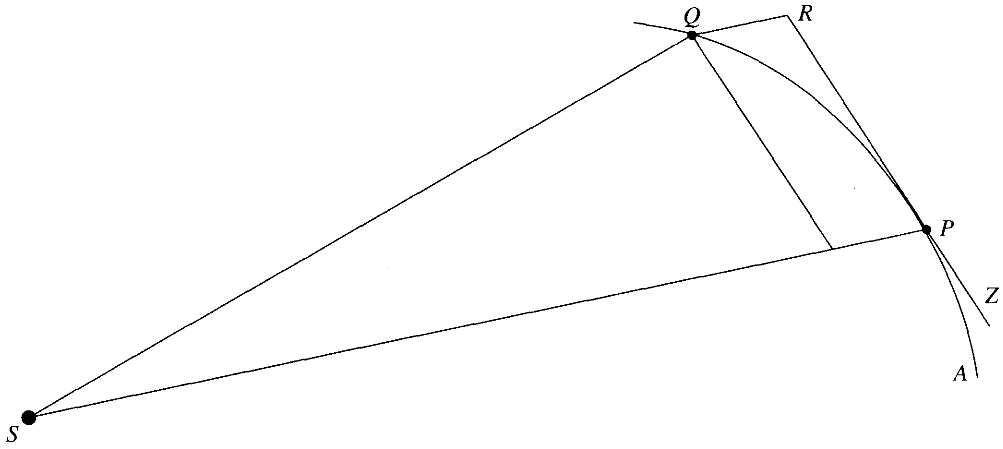


Figure 7.9 (Newton)

ct_2. Since $t_1 = ct_2$, we know that $A_1 = cA_2$. Therefore,

$$\frac{A_2}{t_2} = \frac{cA_2}{ct_2} = \frac{A_1}{t_1},$$

as asserted. So $\frac{A_t}{t}$ is the same constant for any t. We will call this constant the *Kepler constant* of the orbit and denote it by κ (the Greek letter kappa).

7.2 Analyzing Centripetal Force

Continuing the assumption that a body is being propelled by a single centripetal force, Newton begins to turn his attention to the connection between the magnitude of the force and the shape of the orbit.

Let the body be in a typical position P and let F_P be the magnitude of the centripetal force at P. In order to study the centripetal force at P, it is necessary to study its dynamics *near* P. To understand why this is so, just observe that basic properties of a curve at a particular point, for example the slope of the tangent line, can only be assessed by studying the "flow" of the curve near the point.

Again let Δt be a very small interval of time, and suppose that the body has moved from P to Q during Δt. (See Figure 7.9.) During the body's motion from P to Q, the centripetal force will not vary much, and Newton takes it to be constant. So its magnitude is equal to F_P throughout the motion from P to Q. Consider the tangent RPZ to the orbit at P, choosing R such that QR is parallel to the line connecting P to the center of force S. Complete QR and RP to the parallelogram shown in the figure. Newton now imagines the motion of the body from P to Q to be separated into two parts:

(1) The motion from P to R. This part takes place along the tangent line at P. The velocity is constant and equal to the velocity of the body at P. This part of the motion has zero acceleration and is free of the action of the centripetal force.

(2) The motion from R to Q. This part takes place in the direction from P to S and has zero initial velocity. This is the accelerated part of the motion. It is entirely determined by the centripetal force.

As the body proceeds from P to Q these two motions take place simultaneously. But by regarding the motion from P to Q as occurring in the two parts separately, Newton is able to isolate the effect of the centripetal force and can proceed with its analysis.

Let the mass of the body be m units. Since force = mass × acceleration, the force F_P will provide the body with a constant acceleration of $\frac{1}{m}F_P$ for the

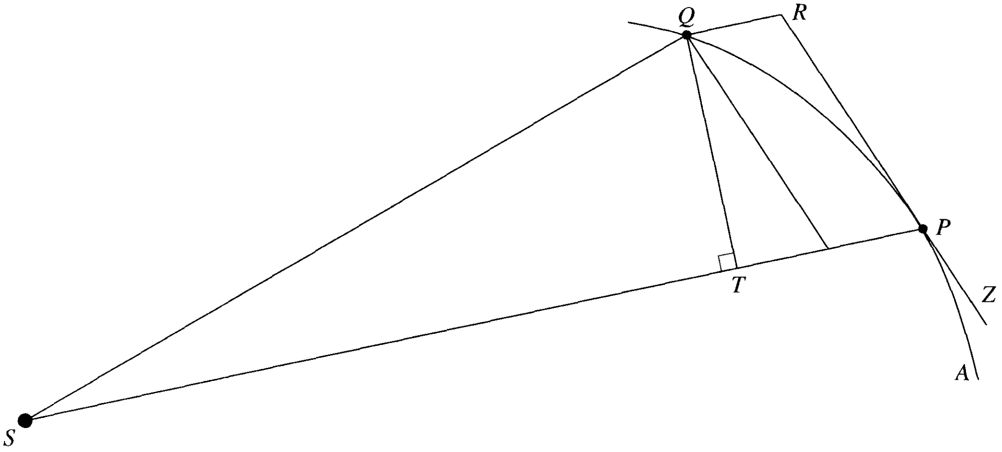


Figure 7.10 (Newton)

duration Δt of its action from R to Q. Now refer to Example 6.14 of Section 6.4 for the following fact. A body that starts from rest and moves with constant acceleration $k = \frac{1}{m}F_P$ will, after a time t, have a velocity of

$$v(t) = kt + v(0) = \frac{1}{m}F_P t + 0 = \frac{1}{m}F_P t$$

and will cover a distance of

$$p(t) - p(0) = \frac{k}{2}t^2 = \frac{1}{2m}F_P t^2.$$

It follows that F_P produces a displacement of $\frac{1}{2m}F_P(\Delta t)^2$ in the body during the time Δt. Therefore, $\frac{1}{2m}F_P(\Delta t)^2 = QR$. So Newton has determined that

$$F_P = \frac{2m\,QR}{(\Delta t)^2}.$$

As in earlier situations, observe that the above equality is only an approximation, but that it improves when Δt is taken smaller.

Newton now expands the diagram in Figure 7.9 to that in Figure 7.10 and makes the following assertion:

COROLLARY I. *If a body P, revolving about the centre S, describes APQ and a right line ZPR touches that curve at any point P; and, from any other point Q of the curve, QR is drawn parallel to the distance SP, meeting the tangent in R; and QT is drawn perpendicular to the distance SP; the centripetal force will be reciprocally as the solid* $\frac{SP^2 \times QT^2}{QR}$; *if the solid is taken of that magnitude which it ultimately acquires, supposing the points P and Q continually to approach to each other.*

What Newton does here is this. He drops the perpendicular QT from Q to SP. Since Δt is small, and hence Q is close to P, he observes that the area of the sector SPQ is in essence equal to that of the triangle SPQ. So, Area sector SPQ = Area ΔSPQ = $\frac{1}{2}(SP \times QT)$. Since SP traces out the sector SPQ in the time interval Δt, it follows from the definition of Kepler's constant κ that

$$\frac{\frac{1}{2}(SP \times QT)}{\Delta t} = \frac{\text{Area sector } SPQ}{\Delta t} = \kappa.$$

So, $\frac{1}{\Delta t} = \frac{2\kappa}{SP \times QT}$. Inserting $\frac{1}{(\Delta t)^2} = \frac{4\kappa^2}{(SP \times QT)^2}$ into the formula $F_P = \frac{2m\,QR}{(\Delta t)^2}$, Newton concludes that

$$F_P = 8\kappa^2 m \frac{QR}{QT^2 \times SP^2}.$$

In particular, F_P is inversely proportional to $\frac{QT^2 \times SP^2}{QR}$, as he asserts in his Corollary I.

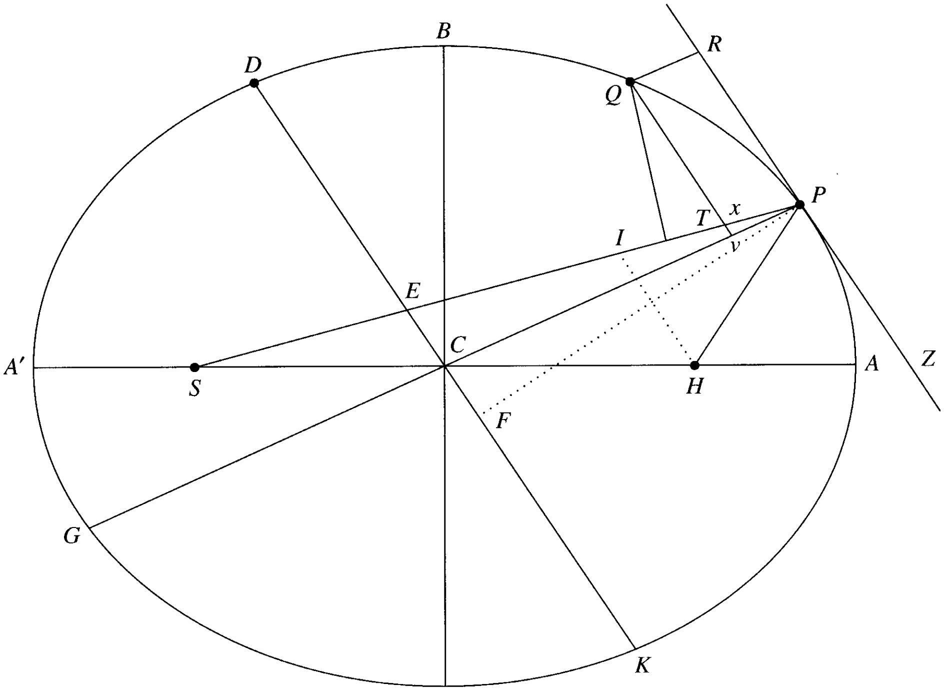


Figure 7.11 (*Newton*)

Consider the rewritten version

$$\boxed{F_P = \frac{8\kappa^2 m}{SP^2}\,\frac{QR}{QT^2}}$$

of this equality. Since the area of ΔSPQ is only an approximation of that of the sector SPQ, the above expression for F_P is only an approximation. However, it improves if Δt is taken to be smaller, in other words, if Q is taken closer to P. For then the area of ΔSPQ approximates that of the sector SPQ with greater precision. Indeed, to obtain an exact formula for F_P it remains to push Q to P and determine what happens to $\frac{8\kappa^2 m}{SP^2}\frac{QR}{QT^2}$ in the process. Since κ and m are constants and SP is the distance from S to P, it is clear that the term $\frac{8\kappa^2 m}{SP^2}$ does not depend on Q. It follows that

$$F_P = \frac{8\kappa^2 m}{SP^2}\cdot\lim_{Q\to P}\frac{QR}{QT^2}.$$

The limit $\lim\limits_{Q\to P}\dfrac{QR}{QT^2}$ is therefore the remaining mystery. Return to Newton's diagram, and notice that the answer to the question of what happens to the ratio $\frac{QR}{QT^2}$ when Q is pushed to P depends entirely on the shape of the orbit between P and Q. At this point therefore, Newton must assume something about the orbit.

7.3 The Inverse Square Law

With later applications in mind, Newton assumes that the orbit is an ellipse and that the center of force S is one of its focal points. Now he can come to grips with the limit $\lim\limits_{Q\to P}\frac{QR}{QT^2}$. What follows is a very subtle argument that is surely one of the most famous in all of science. The diagram on which it is based—see Newton's Figure 7.11—was depicted on the British one-pound note of the 1960s.

Focus on this diagram. The points S, P, Q, R, Z, and T have the same meaning as in Figure 7.10. In addition, C is the center of the ellipse, $A'A$ is the major axis, CB the upper part of the minor axis, DK the diameter of the ellipse parallel to the tangent RPZ, PCG is the indicated diameter, x is the intersection of PS and the parallel to RP through Q, and v is the intersection of this parallel with PCG. Note that $A'A = 2AC$ with $AC = a$ the semimajor axis, and that $CB = b$ is the semiminor axis. Newton explains the rest of his diagram in the course of his argument.

PROPOSITION XI. PROBLEM VI. *Let a body revolve in an ellipsis: it is required to find the law of the centripetal force tending to the focus of the ellipsis.*

Let S be the focus of the ellipsis. Draw SP, cutting the diameter DK of the ellipsis in E, and the ordinate Qv in x; and let the parallelogram QxPR be completed. It is evident that EP is equal to the greater semi-axis AC: for, drawing HI from the other focus H of the ellipsis, parallel to EC, because CS, CH are equal, ES, EI will be also equal; so that EP is the half sum of PS, PI, that is, (because of the parallels HI, PR, and the equal angles IPR, HPZ) of PS, PH; which taken together are equal to the whole axis 2AC.

Newton arrives at these conclusion as follows. Since CS and CH are equal and the triangles ΔSIH and ΔSEC are similar, $ES = EI$. Since $EP = PI + EI$, $PS = PI + 2EI$. So $PS + PI = 2(PI + EI) = 2EP$. Therefore, $EP = \frac{1}{2}(PS + PI)$. Since IH and RZ are parallel, $\angle PHI = \angle HPZ$ and $\angle RPS = \angle PIH$. By a basic property of the ellipse known to the Greeks, $\angle HPZ = \angle RPS$. See Figure 7.12 and Proposition 3.4 of Section 3.1. Therefore, $\angle PHI = \angle PIH$, and hence ΔHPI is isosceles. By properties of isosceles triangles, $PI = PH$. Therefore, $EP = \frac{1}{2}(PS + PH)$. That $PS + PH = AC + A'C = 2AC$ follows from the "string" construction of the ellipse discussed in Exercises 4B of Chapter 4. So as Newton had asserted,

$$EP = AC. \tag{7a}$$

Newton next observes that the triangles ΔPCE and ΔPvx are similar. So $\frac{Px}{Pv} = \frac{EP}{PC}$. Since $RQxP$ is a parallelogram, $QR = Px$. Therefore, $\frac{QR}{Pv} = \frac{EP}{PC}$. Using (7a), Newton has shown that

$$\frac{QR}{Pv} = \frac{AC}{PC}. \tag{7b}$$

Now Newton makes use of another classical property of the ellipse. Let V be any point on a diameter GP of an ellipse. Choose X such that XV is parallel to the tangent at P. According to a classical and basic property of the ellipse, see Figure 7.13 and Proposition 5 of Section 3.1, the ratio $\frac{GV \times PV}{XV^2}$ is the same, no matter which point V is chosen on the diameter GP. Applying this fact first with $V = v$ and again with $V = C$, Newton has shown that $\frac{Gv \times Pv}{Qv^2} = \frac{PC^2}{CD^2}$. Therefore,

$$\frac{1}{Qv^2} = \frac{PC^2}{CD^2} \frac{1}{Gv \times Pv}. \tag{7c}$$

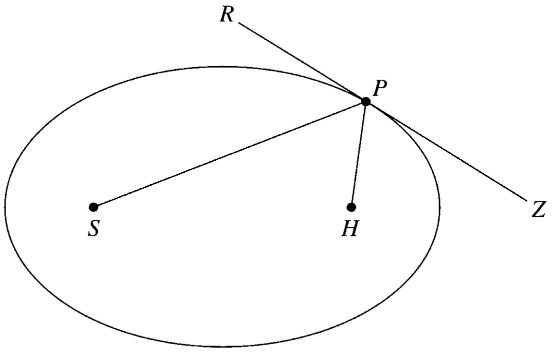


Figure 7.12

Newton next constructs PF perpendicular to DK. Since the segments Qv and EF are parallel, $\angle QxE = \angle PEF$. So the right triangles ΔQTx and ΔPFE are similar, and $\frac{Qx}{QT} = \frac{EP}{PF}$. Therefore, by (7a), $\frac{Qx^2}{QT^2} = \frac{EP^2}{PF^2} = \frac{AC^2}{PF^2}$. For Q very close to P, he can take $Qv = Qx$. So $Qv^2 = Qx^2$, and $\frac{Qv^2}{QT^2} = \frac{AC^2}{PF^2}$. Therefore,

$$\frac{1}{QT^2} = \frac{AC^2}{PF^2} \frac{1}{Qv^2}. \tag{7d}$$

By yet another classical property of the ellipse, the areas of any two parallelograms constructed by

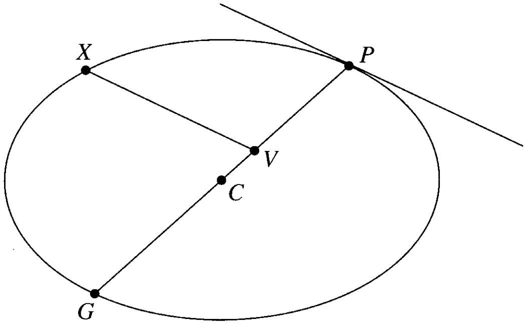


Figure 7.13

starting with any diagonal and taking the indicated tangents are equal. Refer to Figure 7.14. Applying this first to the diagonal DK and then to the diagonal AA', it follows (recall that the area of a parallelogram is equal to base times height) that $2CD \times PF = 2AC \times CB$. Therefore, $\frac{AC}{PF} = \frac{CD}{CB}$, and

$$\frac{AC^2}{PF^2} = \frac{CD^2}{CB^2}. \tag{7e}$$

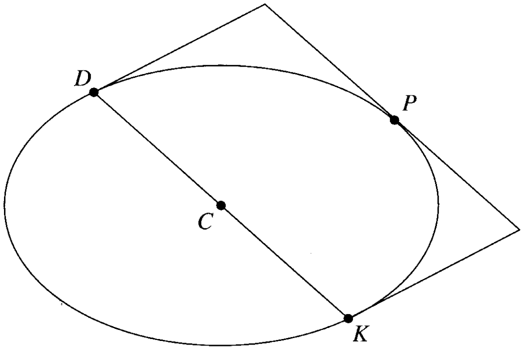


Figure 7.14

Newton has arrived at the last part of his computation. He lets $L = \frac{2CB^2}{AC} = \frac{b^2}{a}$. This quantity is known as the principal latus rectum of an ellipse. He then continues as follows: *and compounding all those ratios together, $L \times QR$ is to QT^2 as $AC \times L \times PC^2 \times CD$, or $2CB^2 \times PC^2 \times CD^2$, to $PC \times Gv \times CD^2 \times CB^2$, or as $2PC$ to Gv. But, the points Q and P continually approaching without end, $2PC$ and Gv are equal. Therefore the quantities $L \times QR$ and QT^2, proportional to these, are also equal. Let these equals be multiplied into $\frac{SP^2}{QR}$, and $L \times SP^2$ will become equal to $\frac{SP^2 \times QT^2}{QR}$. Therefore (by Cor. I. and 5. Prop. VI) the centripetal force is reciprocally as $L \times SP^2$, that is, reciprocally in the duplicate ratio of the distance SP. Which was to be found.*

Let's elaborate on this argument. Newton inserts both $L = \frac{2CB^2}{AC}$ and (7d) into $\frac{L \times QR}{QT^2}$, and then uses (7e), (7c), and (7b), to get

$$\begin{aligned}\frac{L \times QR}{QT^2} &= \frac{2CB^2}{AC} \cdot QR \cdot \frac{AC^2}{PF^2} \cdot \frac{1}{Qv^2}\\ &= \frac{2CB^2}{AC} \cdot QR \cdot \frac{CD^2}{CB^2} \cdot \frac{PC^2}{CD^2}\frac{1}{Gv \times Pv}\\ &= \frac{2}{AC} \cdot QR \cdot PC^2 \cdot \frac{1}{Gv \times Pv}\\ &= \frac{2}{AC} \cdot AC \cdot PC^2 \frac{1}{Gv \times PC}\\ &= \frac{2PC}{Gv}.\end{aligned}$$

Therefore, $\frac{QR}{QT^2} = \frac{2PC}{Gv}\frac{1}{L}$. When Q is pushed to P, $\frac{1}{L}$ does not change, and Gv closes in on $2PC$. Therefore,

$$\lim_{Q \to P} \frac{QR}{QT^2} = \frac{1}{L}.$$

Newton has arrived at his goal. Inserting this result into the formula at the end of Section 7.2, he gets

$$F_P = \frac{8\kappa^2 m}{L} \cdot \frac{1}{(SP)^2}.$$

So Newton has proved[2] the following: Suppose a body is propelled by a single centripetal force and that its orbit is an ellipse with the center of force S at one of the focal points. Then the magnitude F_P of the centripetal force at any point P in its orbit is given by the *inverse square law*

$$F_P = \frac{8\kappa^2 m}{L}\frac{1}{r_P^2},$$

where κ is Kepler's constant of the orbit, m is the mass of the body, $L = \frac{2CB^2}{AC} = \frac{2b^2}{a}$ is the latus rectum, a and b are the semimajor and semiminor axes of the ellipse, and $r_P = SP$ is the distance from P to S.

Let T be the time it takes the body to complete one revolution about S, i.e., let T be the *period* of the orbit. Since the area of the ellipse with semimajor axis a and semiminor axis b is $ab\pi$ (this was verified in Section 4.6), it follows that Kepler's constant is equal to $\kappa = \frac{ab\pi}{T}$. Therefore, Newton's formula can be written as

$$\boxed{F_P = \frac{4\pi^2 a^3 m}{T^2}\frac{1}{r_P^2}}$$

where, to repeat, m is the mass of the body and r_P is the distance from the body to the center of force S. Note that both F_P and r_P vary with time, but that everything else is constant.

Is the orbit of every body moving in the solar system an ellipse? No! When a comet passes appropriately close to a larger planet—Jupiter is a prime example—the gravitational pull of the planet will deflect the comet along a hyperbolic arc or (under certain ideal circumstances) a parabolic arc until this pull becomes negligible with increased distance. Newton addresses such a possibility, again in a completely abstract setting.

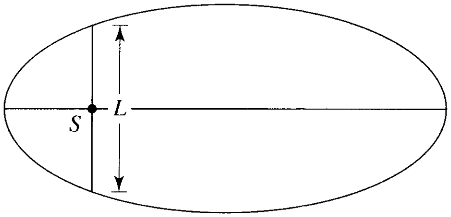


Figure 7.15

It turns out that the latus rectum L of an ellipse is equal to the length of the segment through a focus and perpendicular to the major axis (Figure 7.15). See Exercise 9. The latus rectum L of a parabola or hyperbola can be defined in the same way, as the length of the segment through the focus and perpendicular to the axis (Figure 7.16). Newton shows that the centripetal force on a body of mass m is given by the inverse square law

$$\boxed{F_P = \frac{8\kappa^2 m}{L}\frac{1}{r_P^2}}$$

also if its orbit is a parabola or a hyperbola. As in the case of the ellipse, the verification depends on the fact that $\lim\limits_{Q\to P}\frac{QR}{QT^2} = \frac{1}{L}$.

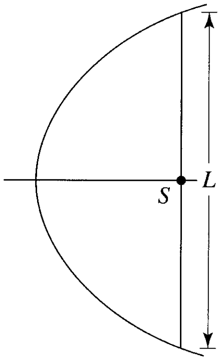


Figure 7.16

Newton next considers the problem in the other direction. If a body moves under the action of a centripetal force with the property that its magnitude is inversely proportional to the square of the distance of the body from the center of force, is the orbit necessarily a conic section, i.e., an ellipse, a parabola, or a hyperbola, and is the center of force at a focus? In particular, if a body subject to such a centripetal force is a closed curve, does this curve have to be an ellipse? The *Principia's* PROPOSITION XVII answers these questions in the affirmative. Newton reasons in the following way. The body in question will, at some fixed point in time t_0, be in a certain position and move in a certain direction with a certain speed. Newton hypothetically considers a second body with the same mass and subject to the same centripetal force as the first that is in elliptical, parabolic, or hyperbolic orbit about the center of force. By setting the various parameters, he obtains the situation where this second body has, at time t_0, the same position and velocity as the first. Since the essential data determining the motion of the two bodies (force, mass, position, and velocity) are the same at time t_0, their orbits must be the same from time t_0 onward. Therefore, the path of the first body is either an ellipse, a parabola, or a hyperbola, at least from time t_0 on. In this way, Newton demonstrates, in particular, that Kepler's first law of elliptical orbits is a mathematical consequence of a more basic law: his inverse square law.[3]

Our look at some of the most important elements of Book I of the *Principia* is now complete. Let's summarize what Newton has accomplished:

Suppose that a body P is propelled by a centripetal force that has center of force S. See Figure 7.17.

Conclusion A. The segment SP sweeps out equal areas in equal times. In particular, if A_t is the area swept out by SP during some time t, then $\frac{A_t}{t}$ is the same constant κ, no matter what t is equal to and no matter where in the orbit this occurs.

Conclusion B. If it is assumed in addition that the orbit of the body is either an ellipse, a parabola, or a hyperbola, then the magnitude F_P of the centripetal force is given by the equation

$$F_P = \frac{8\kappa^2 m}{L}\frac{1}{r_P^2},$$

where m is the mass of the body, L is the latus rectum of the orbit, and r_P is the distance between P and S. Finally,

Conclusion C. If throughout its orbit, the magnitude F_P of the centripetal force is given by an inverse square law, in other words, by an equation of the form $F_P = c\frac{1}{r_P^2}$, where c is a constant, then this orbit is either an ellipse, a parabola, or a hyperbola, and the center of force S is at a focal point.

In the remaining sections of this chapter, we turn to Newton's primary purpose: The application of his abstract mathematical theory of centripetal forces to the explanation of the workings of the solar system. This is the aim of Book III of the *Principia*, the *System of the World.*

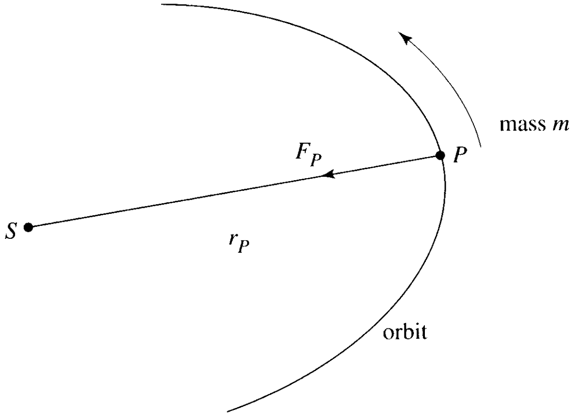


Figure 7.17

7.4 Test Case: The Orbit of the Moon

To demonstrate that his conclusions apply to the real world, Newton must test them against the existing evidence. He begins this task by verifying that very basic observations about the Moon's orbit around the Earth are consistent with his theory.

Let's begin with some numerical data about the Moon's orbit. By Newton's time, these were more accurate than the earlier estimates achieved by the Greeks. The radius of the Earth was understood to be about $R = 4000$ miles, and the distance from the Earth to the Moon about 240,000 miles, or $60R$. It was known that the Moon completes an orbit in 27 days, 7 hours,

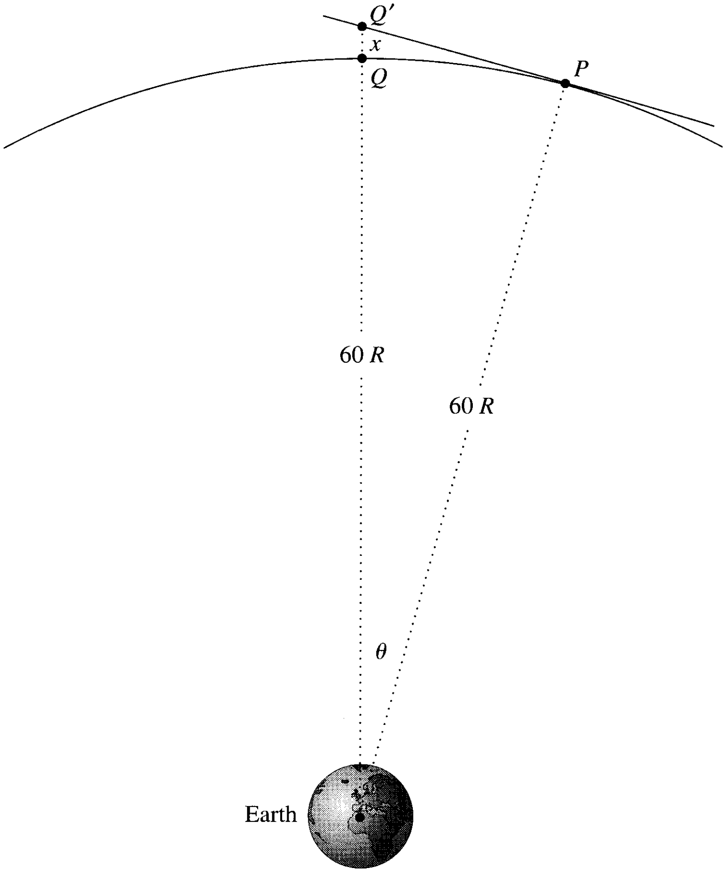


Figure 7.18

and 43 minutes, or 39,343 minutes. We will make the simplifying assumption that the Moon's orbit is a circle of radius $60R$ and that it traces out its orbit at constant speed. Consider the Moon in a typical position P and let it be at Q exactly 1 minute later. In Figure 7.18, the motion of the Moon from P to Q is decomposed into the tangential component PQ' and the component $Q'Q$ in the direction of the Earth. Observe that the angle θ is equal to $\frac{360}{39343}$ degrees. By trigonometry, $\frac{60R}{x+60R} = \cos\theta = 0.999999987$. So $\frac{x+60R}{60R} = 1.000000013$. Solving for x, we get $x+60R = (1.000000013)60R$, and therefore

$$x = (1.000000013 - 1)60R = 0.000000078\,R.$$

This estimate for the distance of the "fall" of the Moon towards the Earth in 1 minute is a consequence of observational data alone. Is the value provided by Newton's theory at least approximately the same?

Consider Newton's equation $F_P = \frac{4\pi^2a^3m}{T^2}\frac{1}{r_P^2}$ as applied to the gravitational pull of the Earth on the Moon. Since the orbit is assumed to be a circle of radius $60R$, the semimajor axis a and the distance r_P are both equal to $60R$. Since $T = 39{,}343$ minutes, we get

$$F_P = \frac{4\pi^2(60R)^3m}{(39{,}343)^2}\,\frac{1}{(60R)^2} = \frac{4\pi^2 60Rm}{(39,343)^2}.$$

This force provides the component of the Moon's motion from Q' to Q with an acceleration of

$$\frac{F_P}{m} = \frac{4\pi^2 60R}{(39{,}343)^2} = \frac{4(9.869604401)(60)}{(39{,}343)^2}R$$
$$= 0.00000153R.$$

Notice that the initial velocity of this motion is zero. An application of Example 6.14 of Section 6.4 shows that after t minutes, the velocity of the Moon and the distance of its "fall" towards the Earth are respectively

$$v(t) = (0.00000153R)t \quad \text{and}$$
$$p(t) = \frac{1}{2}(0.00000153R)t^2.$$

In the current situation, $t = 1$. So the predicted distance of the fall is

$$x = p(1) = \frac{1}{2}(0.00000153R) = 0.000000765R.$$

Taking $R = 4000$ miles and 1 mile $= 5280$ feet, we can calculate that the observed value of the distance per minute the Moon falls towards the Earth is $x = 16\frac{1}{2}$ feet per minute and that the theoretical value is $x = 16\frac{1}{6}$ feet per minute.

Newton's theory has passed the test. The agreement between observation and theory is good. The discrepancy that does exist can be explained by the fact that simplifying assumptions were made. After all, the Moon's orbit was taken to be circular and not elliptical, and the gravitational effects of the Sun were ignored.

7.5 The Law of Universal Gravitation

Newton now has evidence that his inverse square law $F_P = \frac{8\kappa^2 m}{L}\frac{1}{r^2}$ provides a valid quantitative description of the gravitational force with which a body S attracts a body P of mass m that is a distance r from S. He is convinced that his theory applies not only to the orbit of the Moon about the Earth, but also to the orbits of the planets around the the Sun and to the motion of the moons of Jupiter. Indeed, he believes that it is valid anywhere in the universe!

Consider any two bodies S and P in the universe with P in orbit around S. Rewrite the expression for the magnitude of the force on P as

$$F_P = C_P m \frac{1}{r^2}$$

where C_P is the constant $\frac{8\kappa^2}{L}$. If S exerts a pull on P, why should not also P exert a pull on S? Indeed, by Newton's third law, the force F_P with which S pulls on P has an equal and opposite reaction. In other words, P pulls on S with a force of magnitude $F_S = F_P$ in the opposite direction of that with which S pulls on P. See Figure 7.19. For instance, a planet attracts the Sun with a force equal to that which the Sun exerts on the planet. The effects of the two forces are different, of course, because the Sun's mass is much greater than that of the planet.

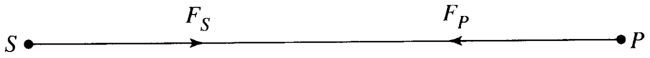


Figure 7.19

The symmetry of the situation requires that F_S can be expressed as

$$F_S = C_S M \frac{1}{r^2},$$

where M is the mass of S and C_S is a constant. In the *System of the World*, Newton puts the matter this way:

> Since the action of the centripetal force upon bodies attracted is, at equal distances, proportional to the quantities of matter in those bodies, reason requires that it should be also proportional to the quantity of matter in the body attracting.

Since $F_S = F_P$, we see that $C_S M = C_P m$ and hence that $\frac{C_P}{M} = \frac{C_S}{m}$. Now let $G = \frac{C_P}{M} = \frac{C_S}{m}$ and notice that $GM = C_P$ and $Gm = C_S$. Let $F = F_P = F_S$, and substitute to get

$$\boxed{F = G\frac{mM}{r^2}}$$

This is Newton's *Law of Universal Gravitation.* It asserts that any two bodies in the universe—whether one is in orbit about the other or not—attract each other with a force that is given by the above formula. Newton recognizes that the masses m and M of the two bodies and the distance r between them are the essential parameters determining the force. He recognizes, in particular, that G should be a *universal constant,* in other words, a constant that is the same for any two bodies, separated by any distance, anywhere in the universe.

This reciprocal nature of gravitational attraction has important consequences for the motion of the bodies in the solar system. As Newton explains:

> I have hitherto explained the motions of bodies attracted towards an immoveable centre, though perhaps no such motions exist in nature. For attractions are made towards bodies; and the actions of bodies attracting and attracted are always mutual and equal, by the third law of motion: so that, if there are two bodies, neither the attracting nor the attracted body can really be at rest; but both as it were by a mutual attraction, revolve about the common center of gravity.

To understand what Newton is saying, consider the Earth–Moon system for example. The center of mass (or gravity) of this system, this is the so-called *barycenter* of the system, is about 2900 miles from the center of the Earth, or, about 1050 miles below its surface. The centers of mass of both the Moon and the Earth travel along ellipses with the barycenter at a focus of each ellipse. To understand what is going on, think of the centers of the Earth and Moon as being connected with a horizontally placed lever with fucrum at the barycenter **B**. See Figure 7.20 and note that the lever is balanced. Now let the lever revolve in the horizontal plane. This simulates the dynamics of the Earth–Moon system. The Moon is in a month-long orbit around **B**. This is a circular orbit in the simulation, but elliptical in fact. The center C of the Earth is also in "orbit" around **B**. In other words, the Moon's gravitational pull on the Earth causes it to wobble about its center C in a monthly cycle, as indicated in Figure 7.21.

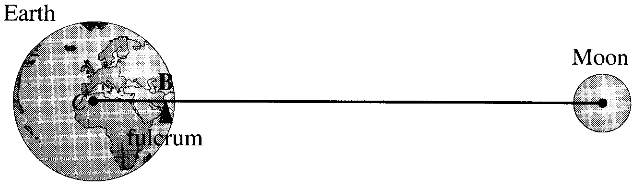


Figure 7.20

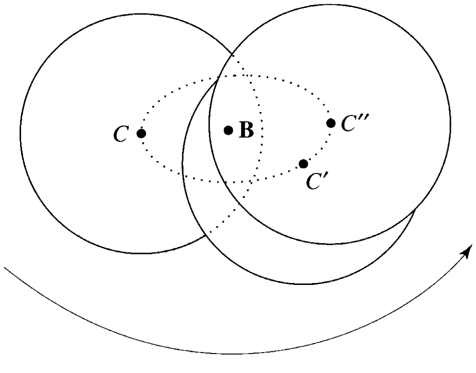


Figure 7.21

Let's apply these considerations to the solar system as a whole. It is the barycenter **B** of the Earth–Moon system (rather than the center of the Earth) that describes an elliptical orbit about the Sun. The center of force is not the Sun, but rather the center of mass of the entire system of the Sun and planets.

This point is at a focus of the ellipses of all the planets. Newton's explanations thus constitute a subtle refinement of Kepler's laws. Since the Sun comprises 99.85% of the mass of the solar system, the difference is only minor, however. In reference to the orbit of the revolving Earth–Moon system about the Sun, observe that the center of the Earth is at times ahead of the Moon and at times behind it.[4] Note also that when the Earth is closer to the Sun, the gravitational tug of the Sun on the Earth–Moon system will be greater than when the Moon is closer to the Sun (because the Earth's mass is greater than that of the Moon). The Sun's gravitational force on the Earth–Moon system thus depends on the dynamics of this system. (Section 14.5 considers subtleties about gravity that are related to the point just made.) It should now be apparent that the mathematics of the "three-body problem" of the gravitational interaction of the Sun, Earth, and Moon is very complicated[5] (and still awaits a complete solution).

These considerations explain other phenomenona as well. The shape of the Earth is essentially spherical, but it is a sphere that is flattened at the poles. This flattening was brought about by the rotation of the Earth about its axis. Consider the plane through the Earth's equator. Since the Moon is near this plane at every point in its orbit (the Moon's orbital plane and the Earth's equatorial plane only differ by up to 5°), it follows that the Moon's gravitational tug on the Earth is a little greater at the equator than at the poles. The difference is substantial enough to produce a very slow gyration of the Earth's axis of rotation. (Why is the analogous effect of the Sun's gravitational force on the Earth negligible?) Translated to the Earth-centered point of view of Figure 2.1 of Section 2.1, this gyration is equivalent to a slight rotation of the axis that joins the two equinox positions. This "precession of the equinoxes" was already observed by the Greeks. A similar effect explains the tides of the oceans: they are produced by the tug of the Moon on the Earth's surface.

We turn next to some of the incredible implications of Newton's Laws of Universal Gravitation.

7.6 Incredible Consequences

Consider the situation of a body P in elliptical orbit about another body S. Suppose that S is much more massive than P. Therefore, in the context of the explanations just given, S is in essence the center of the centripetal force, and the mass M of S is the same as that of the system. Recall that Kepler's constant for the elliptical orbit is $\kappa = \frac{\pi ab}{T}$ and the latus rectum is $L = \frac{2b^2}{a}$, where as before, a is the semimajor axis, b the semiminor axis, and T the period of the orbit. Combining this with the equations

$$C_P = \frac{8\kappa^2}{L} \quad \text{and} \quad C_P = GM$$

from Section 7.5 shows that $GM = \frac{8\kappa^2}{L} = \frac{8a\kappa^2}{2b^2} = \frac{4a\pi^2a^2b^2}{b^2T^2} = \frac{4\pi^2a^3}{T^2}$. Therefore,

$$\boxed{\frac{a^3}{T^2} = \frac{GM}{4\pi^2}}$$

Notice that the term $\frac{GM}{4\pi^2}$ on the right has nothing to do with the particulars of the body P and its orbit. In other words, it is the same for any P in orbit around S. It follows that the ratio $\frac{a^3}{T^2}$ of the cube of the semimajor axis a to the square of the period T of the orbit is the same for any body P in orbit around S. In the case of the planets of the solar system, this is precisely what Kepler's third law asserts. In other words, Kepler's third law is a consequence of Newton's theory of gravitation! Refer to the conclusions A and C of Section 7.3, and observe therefore that all three of Kepler's laws are consequences of Newton's theory.

Consider the rewritten version

$$\boxed{M = \frac{4\pi^2a^3}{GT^2}}$$

of the previous equation. If, somehow, the universal constant G could be determined, then the mass of the Sun could be computed by inserting a and T for the orbit of the Earth!! In the same way, the mass of the Earth could be computed by inserting data about the orbit of the Moon. Newton, however, seems to think that the determination of G is out of reach:

> Perhaps it may be objected, according to this philosophy all bodies should mutually attract one another, contrary to the evidence of experiments in terrestrial bodies. But I answer, that the experiments in terrestrial bodies come to no account. For the attraction of homogeneous spheres near their surface are as their diameters. Whence a sphere of one foot in diameter, and of like nature to the Earth, would attract a small body placed near its surface with a force of 20,000,000 less than the Earth would do if placed near its surface. But so small a force could produce no sensible effect.

In other words, Newton regards the experimental determination of the constant G to be an impossible enterprise. This time Newton was wrong! In the latter part of the 18th century, in 1798 to be exact, Henry Cavendish[6] succeeded in measuring G in the laboratory with an extremely delicate experiment. What he did in essence was this. Suspend a fine wire from a fixed point A and attach a rigid "crossbar" BC. Refer to Figure 7.22. From BC in turn, suspend two heavy iron balls. They are shown in black. Now move two more heavy iron balls (shown lightly shaded) in place near the two others. If this apparatus is very delicately balanced and controlled, then the gravitational force F between the two pairs of balls will bring about a rotation of the axis BC. This allows F to be measured. Since Cavendish knew the masses of the balls and the distances between them, the equation $F = G\frac{mM}{r^2}$ allowed him to compute

$$G = 6.67 \times 10^{-11} \frac{\text{meters}^3}{\text{kilograms} \cdot \text{seconds}^2}.$$

Using the fact that the semimajor axis of the Earth's orbit about the Sun is 149.6×10^6 kilometers, or 149.6×10^9 meters and that its period T is 365.25 days, Cavendish, after expressing T in seconds, could now compute the mass M of the Sun:

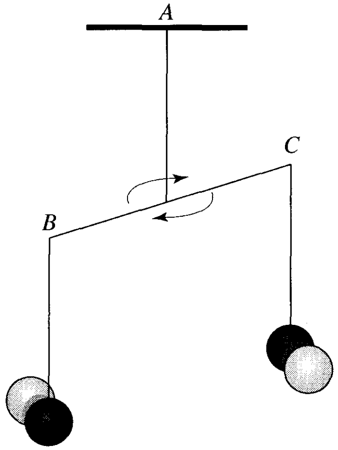


Figure 7.22

$$M = \frac{4\pi^2 a^3}{T^2 G} = \frac{4(3.14)^2(149.6 \times 10^9)^3}{[(365.25)(24)(60)(60)]^2(6.67 \times 10^{-11})}$$

$$\approx \frac{1320 \times 10^{32}}{66 \times 10^3} \approx 2 \times 10^{30} \text{ kg}.$$

The same calculation (making use of information of the Moon's orbit about the Earth) shows that the mass of the Earth is about 6×10^{24} kilograms. Incredibly, Newton's theory of universal gravitation together with Cavendish's value of G has "served up" estimates for the masses of the Sun and the Earth!

Consider, finally, a projectile in motion near the Earth's surface. In view of their universal nature, the above considerations should apply to this situation as well. Do they? Figure 7.23 shows a typical trajectory and typical position P of the projectile and the surface of the Earth. Observe that the projectile is subject to a centripetal force whose center is the center of mass C of the Earth. Neglecting air resistance, we get as a consequence of Newton's theory that the trajectory must be an arc of an ellipse. The parabola of the discussion of Section 6.5 is only an approximation. (What is the difference between the underlying assumptions about the action of gravity?) Now suppose that the projectile is in free-fall and let F_P be the magnitude of the force of gravity. By the universal law of gravitation, $F_P = G\frac{mM}{r^2}$, where M is the mass of the Earth, m the mass of the projectile, and r the radius of the Earth. By Newton's second law, $F_P = mg$, where g is the acceleration produced by the Earth's gravity. By putting the two equations for F_P together, we get

$$g = G\frac{M}{r^2}.$$

After inserting the values $G = 6.67 \times 10^{-11} \frac{\text{meters}^3}{\text{kg}\cdot\text{sec}^2}$, $M = 6 \times 10^{24}$ kg, and $r = 6360$ kilometers (this is equivalent to 3950 miles), the result is

$$\begin{aligned} g &= 6.67 \times 10^{-11} \cdot \frac{6 \times 10^{24}}{(6360 \times 10^3)^2} \\ &\approx 6.67 \times 10^{-11} \cdot \frac{6 \times 10^{24}}{4 \times 10^7 \times 10^6} \approx 10. \end{aligned}$$

The unit is meters/second^2. Note the close agreement with the observed value of $g = 9.80$ meters/second^2. Newton's theory of gravitation has provided the explanation for Galileo's observation that all bodies falling near the Earth's surface are subject to the same acceleration. The fact that his theory is valid not only in "outer space" but also "here at home" is further indication of its universality.

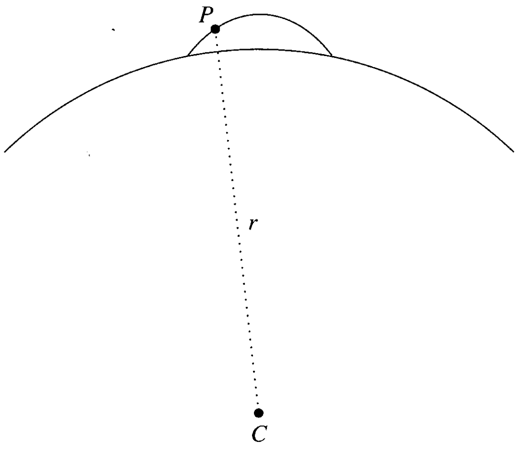


Figure 7.23

7.7 Postscript

Perhaps the greatest triumph of Newton's theory is described by the following sequence of events. In 1781, the British astronomer William Herschel discovered what he originally thought to be a comet, but what later proved to be a planet. The planet Uranus thus joined Mercury, Venus, Earth, Mars, Jupiter, and Saturn—already known since antiquity—and became the seventh planet of the solar system. Efforts were made to compute its orbit, but discrepancies appeared between observed positions and those predicted by theory, and these grew year by year. Several astronomers hypothesized that the gravitational pull of yet another planet on Uranus would explain the discrepancies. By 1845, the British astronomer Adams, and later in 1846, the Frenchman Leverrier, had both computed the hypothetical orbit of the hypothetical planet. Later in 1846, the Berlin Observatory discovered a new planet in a position close to that predicted by these calculations! This eighth planet was later named Neptune. A similar sequence of events led to the discovery of the ninth (and last known) planet Pluto in 1930.

The fact that the planets are attracted not only by the Sun, but also by each other, especially by the more massive Jupiter and Saturn, raises an important concern, of which Newton was certainly aware. Would the effects of the gravitational forces of the planets on each other accumulate over time to threaten the stability of the solar system? Might there be a catastrophic collapse? Consider any of the planets P and let m_P be its mass. It was indeed discovered that neither the semimajor axis a_P of its orbit nor the linear eccentricity e_P nor the angle i_P of inclination of its orbit with its axis of rotation are constants! They are functions of time driven by the gravitational effects of the other planets. The French scientists Lagrange and Laplace carried out a series of investigations in the last part of the 18th century and came to the following remarkable conclusions: The sums

$$(e_{P_1})^2 m_{P_1}\sqrt{a_{P_1}} + (e_{P_2})^2 m_{P_2}\sqrt{a_{P_2}} + \ldots$$

and

$$(i_{P_1})^2 m_{P_1}\sqrt{a_{P_1}} + (i_{P_2})^2 m_{P_2}\sqrt{a_{P_2}} + \dots$$

taken over all the planets are both constants.[7] In other words, while the individual parameters a_P, e_P, and i_P vary, this variation is controlled and limited in the context of the solar system as a whole. So the wobbles and irregularities of the planetary clockwork of the solar system are minor and self-correcting, and do not pose a threat to the stability of the system. End of the story? Hardly! See Section 14.7.

We have reached the end of the first part of this book. It has emerged from our discussion that by thinking about this question and the other, by solving this problem and that, scientists developed both a new outlook and new methods of inquiry. We have traced some of the main themes—especially the mathematical themes—of this *Scientific Revolution*. We saw that this revolution had Greek origins, both from the point of view of the nature of the questions that were asked and the mathematical strategies that were developed in response to them. We saw that it was tied closely to the study of motion, both on the Earth and in the heavens. This scientific revolution reached its culmination in the latter part of the 17th century with the synthesis of astronomy and mechanics and the development of the necessary mathematics, and in particular with Newton's *Principia*. The role played by mathematics in this development was both pervasive and crucial. The explanation of basic physical phenomena by a combination of fundamental laws and mathematics, and the concurrent confirmation of the explanation by observation and experiment, became the basis of modern science.

The success of this outlook and the methods that it spawned energized efforts to extend them to newly emerging fields of science (chemistry, biology, and geology, for example). More generally, however, it contributed to the philosophical underpinnings of an intellectual movement that encouraged the overhaul of every kind of traditional teaching, questioned our entire intellectual heritage, and ultimately brought changes that had significance for the history of humanity in its broadest sense. The transmission of the results of the scientific revolution to the outside world linked the scientific revolution with a new age, the *Enlightenment*. The discoveries of 17th century science were translated into a new outlook and a new world-view, not by scientists themselves, but by a literary movement energized by the *philosophes*. These were radical intellectuals (Voltaire, Rousseau, Diderot, Kant, Locke, Hume,...) who held the secular view that human beings are mature enough to find their own way. Human beings are able to understand the natural world, indeed their own nature, by reason and the methods of mathematics and science. Therefore, they should think for themselves, shake off the hand of authority in the realms of both religion and politics, and insist on the right to freedom of thought and expression. On the commercial side, the philosophes expressed the view that merchants should be free to buy and sell as they chose, that free competition was essential to trade, and that open markets would free commercial energies and benefit everyone. The philosophes made these appeals—as Galileo had tried to do earlier—to a new arbiter of human thought: a wider general reading public.[8]

The *Industrial Revolution*—as epitomized by James Watt's steam engine (invented in 1769) and the power that it provided to machines of production—is also directly linked with the earlier scientific revolution. The spreading criticism of authority, the accelerating pace of the industrialization of the Western world, and the steady increase in population—all pointed to the prospect of instability. Indeed, in the last half of the 18th century, country after country was afflicted by social unrest. The American Revolution and the French Revolution soon followed ... . But it is time to return to mathematics.

Exercises
7A. Equal Areas in Equal Times

1. Suppose that an object P moves along a straight line L with constant velocity. Let S be a point not on L. Regard P to be subject to a certripetal force of zero

magnitude centered at S. Show that the segment SP sweeps out equal areas during equal times. Note the connection between the magnitude of the force and the shape of the orbit.

Exercises 2–5 refer to Newton's proof of Kepler's second law in Section 7.1. Recall that the semimajor axis of the Earth's orbit is $a = 1$ AU. By Example 4.16 of Section 4.7, the semiminor axis is $b = 0.9998$ AU. For purposes of Exercises 2–5 we will therefore assume that the orbit is a circle of radius 1 AU. In Figure 7.24, C is the center of the Earth's orbit and S is the position of the Sun. In addition, P is the position of the Earth when it is nearest the Sun and Q is its position $t_1 =$ two months $= 61$ days later. Let $\Delta t = 1$ day. So P_1 is the position of the Earth one day after it is at P (note that the figure is not drawn to scale). The linear eccentricity is $e = 0.0167$ AU, but take $e = 0.0200$ AU. So $SP = a - e = 0.9800$ AU. Recall also that the Earth completes one orbit in 365.2422 days. Compute with 4 decimal accuracy.

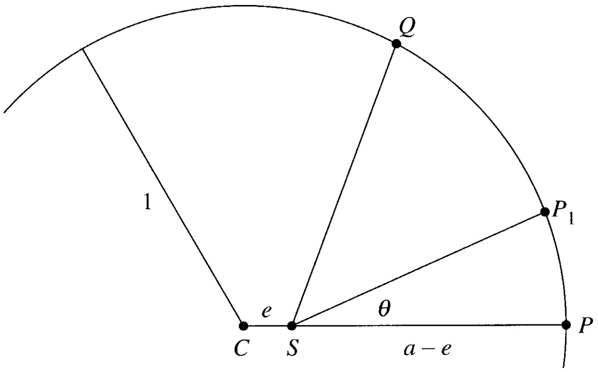


Figure 7.24

2. How many triangles are inscribed into the region SPQ? In other words, what is n equal to in this situation?
3. The area of the Earth's orbit is $\pi \cdot 1^2 = \pi$ in AU^2. Use this to compute Kepler's constant κ for Earth. Making use of κ, compute the area of the region SPQ in AU^2. This is A_1. In the same way, compute the area of the region SPP_1 in AU^2.
4. The region SPP_1 is approximately equal to the circular sector with radius $a - e$ and angle θ. See Figure 7.25. Assume equality and use this and the results of Exercise 3 to compute θ. Show next that the isosceles triangle ΔSPP_1 has area $(a - e)^2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}$ in AU^2. This corresponds to ΔA.

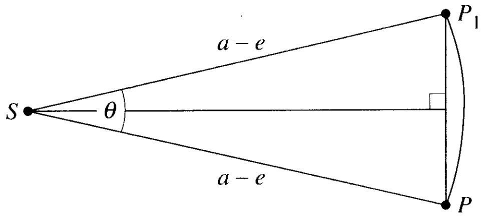


Figure 7.25

5. Compute $n(\Delta A)$ in AU^2 and compare this to A_1. Is $n(\Delta A)$ an acceptable approximation of A_1? Before you answer, convert these areas into square miles using 1 AU $= 93 \times 10^6$ miles.
6. Using the data in Table 4.2 of Section 4.7, compute Kepler's constant for Mercury and Jupiter in AU^2/year.
7. Consider a planet in orbit around the Sun S. Let κ be its Kepler constant. Begin to time the planet when it is at the perihelion position P. This is time $t = 0$. Let $P(t)$ be its position at any time t thereafter, and let $A(t)$ be the area of the region that P, S, and $P(t)$ determine. See Figure 7.26, but allow for the possibility that the planet may already have completed several orbits during time t. Compute the function $A(t)$ in terms of κ.

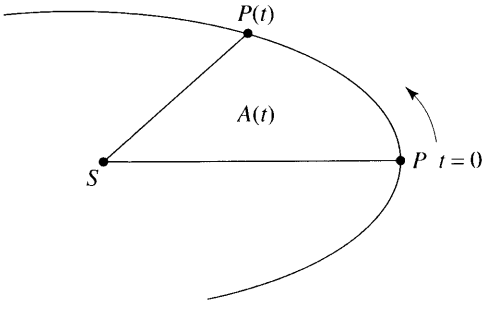


Figure 7.26

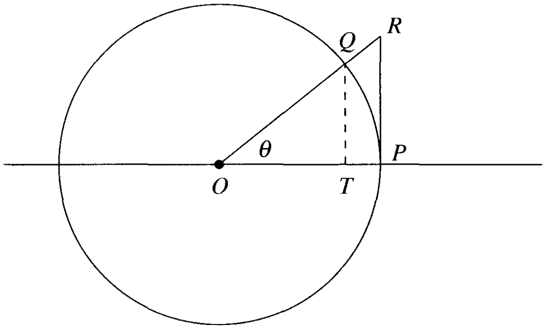


Figure 7.27

7B. Computations Related to the Inverse Square Law

8. Consider Newton's result $\lim_{Q\to P}\frac{QR}{QT^2} = \frac{1}{L} = \frac{a}{2b^2}$ for the ellipse. See Figure 7.27. Give an interpretation of this limit in the case of the circle of radius 1. Write QT in terms of $\sin\theta$ and QR in terms of $\sec\theta$. Then express $\frac{QR}{QT^2}$ in terms of $\cos\theta$. Use the equalities $\sec^2\theta - 1 = \tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$. Now push Q to P, or what is equivalent, θ to 0. Make use of the fact that $\cos\theta = OT$, and hence that $\lim_{\theta\to 0}\cos\theta = \lim_{\theta\to 0} OT = 1$. Does your result agree with Newton's?

9. Consider an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. So a is the semimajor axis and b the semiminor axis. Determine the coordinates of the two focal points in terms of a and b. (A review of some basics from Section 4.5 may be necessary.) Show that the latus rectum $\frac{2b^2}{a}$ is equal to the length of the segment that goes through the focus, is perpendicular to the major axis, and has both endpoints on the ellipse.

7C. The Satellites of Jupiter

10. In the *System of the World*, Newton considers the following data:

$$5.578, \quad 8.876, \quad 14.159, \quad 24.903.$$

They are the maximal distances from Jupiter (as measured by Newton's contemporary the astronomer Flamsteed) of the four satellites of Jupiter discovered by Galileo. Here 1 unit is the radius of Jupiter. The corresponding periods of the orbits of these satellites in hours are respectively

$$42.48, \quad 89.30, \quad 172, \quad 402.09.$$

Both sets of values are from the *System of the World*. Newton checked his theory of gravitation against the data. Which part of his theory did he test, and how good is the fit?

7D. Overview of the Formulas

Inverse Square Law. Suppose that a body of mass m is subject to a centripetal force F_P and that the orbit is an ellipse, a parabola, or a hyperbola. Then

$$F_P = \frac{8\kappa^2 m}{L}\frac{1}{r_P^2},$$

where κ is Kepler's constant, r_P is the distance from the orbiting body to the center of force, and L is the latus rectum. If the orbit is an ellipse, then this formula becomes

$$F_P = \frac{4\pi^2 a^3 m}{T^2}\frac{1}{r_P^2},$$

where a is the semimajor axis and T is the period of the orbit.

Law of Universal Gravitation. The gravitational force F between two bodies is given by

$$F = G\frac{mM}{r^2},$$

where $G = 6.67 \times 10^{-11}\frac{\text{meters}^3}{\text{kilograms}\cdot\text{seconds}^2}$, m and M are the masses of the two bodies in kilograms, and r is the distance between them in meters.

If a body of mass m is in elliptical orbit around another body of mass M, then

$$\frac{a^3}{T^2} = \frac{GM}{4\pi^2} \quad \text{or} \quad M = \frac{4\pi^2 a^3}{GT^2}$$

where a, T, and G are as before.

7E. Systems of Units

Study the following system of units:

Length: 100 centimeters = 1 meter, 1000 meters = 1 kilometer.

Mass: 1000 grams = 1 kilogram. 1 kg is defined to be the mass of 1000 cubic centimeters, i.e., one liter, of water (at 4 degrees centigrade). 1000 kilograms = 1 metric ton.

Force: 1 newton = the (constant) force required to accelerate in 1 second a 1 kilogram mass from rest to a velocity of 1 meter per second.

What is the weight of a mass of 1 kilogram on the surface of the Earth? On the surface of the Earth, weight = mass ×

gravitational acceleration. Therefore,

$$\text{weight} = (1\text{ kilogram})\left(9.80\frac{\text{meters}}{\text{second}^2}\right) = 9.80 \times 1\frac{\text{kg}\cdot\text{m}}{\text{sec}^2}$$

$$= 9.80\text{ newtons.}$$

So on the surface of the Earth, 1 kg weighs 9.8 newtons. The above is the M.K.S. (meter-kilogram-second) system.

The commonly used American system is as follows:

Length: 1 foot, 5280 feet = 1 mile.
Mass: 1 pound.
Force: 1 slug = the mass that a 1 pound force accelerates in 1 second from rest to a velocity of 1 foot per second.

How many pounds does a mass of 1 slug weigh on the surface of the Earth? Again, weight = mass × gravitational acceleration, or

$$\text{weight} = (1\text{ slug})\left(32.17\frac{\text{feet}}{\text{second}^2}\right) = 32.17 \times 1\frac{\text{slug}\cdot\text{feet}}{\text{sec}^2}$$

$$= 32.17\text{ pounds.}$$

The connection between the American system and the M.K.S. system is given by:

Length: 1 foot = 0.30 meters, 1 meter = 3.28 feet, 1000 meters = 1 kilometer = 0.62 miles, 1 mile = 1.61 kilometers.
Mass: 1 pound = 4.45 newtons, 1 newton = 0.22 pounds.
Force: 1 slug = 14.59 kilograms, 1 kilogram = 0.07 slugs.

11. Recall from Table 1.4 of Section 1.5 that the radius of the Earth is 3950 miles and that of the Moon is 1080 miles. Show that this corresponds to 6360 kilometers for the Earth and 1740 kilometers for the Moon.

12. For a basketball to satisfy regulations, its weight must fall between $1\frac{1}{4} = 1.25$ and $1\frac{3}{8} = 1.375$ pounds. (Information supplied by Rawlings Sporting Goods.) What is the mass in slugs of a basketball that weighs 1.3 pounds?

13. Convert the universal gravitational constant $G = 6.67 \times 10^{-11}\frac{(\text{meters})^3}{\text{kilograms}(\text{seconds})^2}$ to $\frac{(\text{feet})^3}{\text{slugs}(\text{seconds})^2}$.

7F. Applying Newton's Formulas to the Moon

14. Show that the mass of the Earth is 6.0×10^{24} kilograms by using information about the Moon's orbit. Use Exercise 11 to compute the average density of the Earth in kilograms per cubic meter.

15. Show that the mass of the Moon is 12.0×10^{22} kilograms under the assumption that its density is the same as that of the Earth. Use the fact that its radius is 1740 kilometers. [Note: A better estimate for the mass of the Moon is 7.4×10^{22} kg. The Moon is therefore less dense than the Earth.]

16. Refer to Figure 7.20. Use the fact that the masses of the Earth and the Moon are 6.0×10^{24} kilograms and 7.4×10^{22} kilograms respectively and that the distance from the Earth to the Moon is 240,000 miles in order to estimate the distance in miles from the barycenter **B** of the Earth–Moon system to the center C of the Earth.

17. Gravity provides all bodies falling near the surface of the Moon with the same acceleration regardless of the mass of the body. Use the fact that the mass of the Moon is 7.4×10^{22} kilograms and that its radius is 1740 kilometers to show that this acceleration is about 1.63 meters/second^2. Since 1 meter is 3.28 feet, this is equal to about 5.35 feet/second^2.

18. How much does the basketball of Exercise 12 weigh on the Moon?

19. Return to Galileo, his cannonball, and the 177-foot-high tower of Pisa. Assume that such a tower has been constructed on the Moon. How long would it take for a cannonball to hit the ground if it is dropped with zero initial velocity from this tower? Compare your result with that of Example 6.15 of Section 6.4. [Note: Recall that on the Earth it had to be assumed that there is no air resistance. No assumption of this kind is needed for the Moon because it has no atmosphere.]

20. Make a calculation to show that the gravitational force of the Sun on the Moon is greater than the force of the Earth on the Moon. This being so, how is it that the Moon goes around the Earth and not around the Sun? Or does it? [Note: Use the following data: the mass of the Sun is 2.0×10^{30} kg; the mass of the Earth is 6.0×10^{24} kg; the mass of the Moon is 7.4×10^{22} kg; the average distance from the Earth to the Sun is 1.5×10^{11} meters; the average distance from the Moon to the Earth is 3.8×10^{8} meters.]

7G. Computing Masses and Forces

21. Suppose that the Earth is shrunk to the size of a basketball. Keeping the density constant, compute the mass of this sphere. How many times more massive is it than that of the basketball of Exercise 12? [Hints:

Recall from Exercise 25 of Chapter 1 that the radius of the Earth must be shrunk by a factor of $\frac{1}{50{,}000{,}000} = 2 \times 10^{-8}$ to obtain a sphere of the size of a basketball. This sphere has volume $\frac{1}{50{,}000{,}000^3} = 8 \times 10^{-24}$ of that of the Earth. Use the fact that the Earth has a mass of 6.0×10^{24} kilograms.]

22. On October 5, 1957, Sputnik, the first artificial satellite of the Earth, was launched by the (then) Soviet Union. It was reported to encircle the Earth at an altitude of 560 miles above the Earth's surface, to travel 18,000 miles per hour, and to make one revolution in 95 minutes. Are these data consistent? Assume that the satellite's orbit is a circle. Use these data to estimate the mass of the Earth in slugs. How does this compare to 6.0×10^{24} kilograms?

23. Space probes are used to determine the masses of the planets by studying the orbital data from their hyperbolic "flybys." Draw a sketch and explain what makes such conclusions theoretically possible. [Hint: Use the formulas $F_P = \frac{8\kappa^2 m}{L}\frac{1}{r^2}$ and $F_P = G\frac{Mm}{r^2}$.]

24. Refer to the Exercises of Chapter 4 that deal with Halley's comet. Compute the latus rectum for Halley's comet. Use data from Halley's orbit to compute the mass of the Sun.

25. Consider an object of mass 250 grams. Determine its weight in newtons. This object is twirled on a string of length 80 centimeters. It describes a circle of this radius and revolves at constant angular velocity of 3 revolutions per second. Determine the centripetal force in newtons necessary to keep it in motion.

7H. A Speculation of Newton

Newton makes the following statement in the *System of the World*:

> For the attraction of homogeneous spheres near their surfaces are as their diameters. Whence a sphere of one foot in diameter, and of like nature to the Earth, would attract a small body placed near its surface with a force of 20,000,000 less than the Earth would do if placed near its surface. But so small a force could produce no sensible effect. If two such spheres were distant but by $\frac{1}{4}$ inch, they would not even in spaces void of resistance, come together by the force of their mutual attraction in less than a month's time.

Is what Newton is saying here correct? The circles in Figure 7.28 represent two spheres in a weightless situation. For example, think of them in the cargo bay of an orbiting space shuttle. The spheres are 1 foot in diameter. The radii are thus $\frac{1}{2}$ foot. The spheres are $\frac{1}{8}$ of an inch, or 0.01 feet, from the vertical axis. So $a = 0.01$ and $b = 0.51$. Both spheres are assumed to have the same density as the average density of the Earth. (Use Exercise 14.) What is the mass of each sphere in slugs? What is the gravitational force with which each sphere attracts the other when they are in the indicated position; when they touch each other? Why is the force in the second situation greater? Suppose that this greater force acts on the sphere (say, the one on the right) when it is in the position indicated in the figure. How long will it take for this force to move the sphere so that it touches the vertical axis? Do the same computation with the weaker force. How do your findings mesh with Newton's assertions?

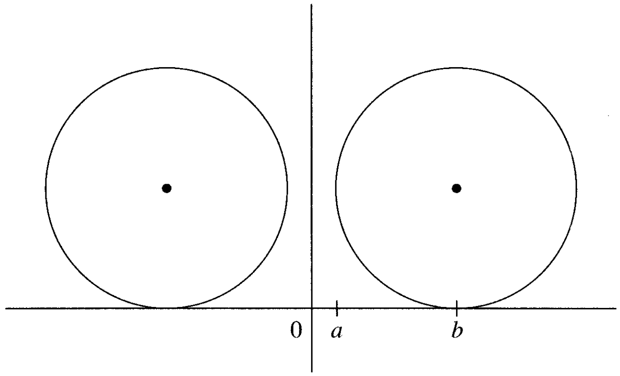


Figure 7.28

Notes

[1] All quotations are taken from Isaac Newton, *Mathematical Principles of Natural Philosophy*, Translated into English and Illustrated with a Commentary by Robert Thorp, London, Printed for W. Strahan and T. Cadell, in the Strand, 1777. Reprinted by Dawsons of Pall Mall, London, 1969.

[2] Observe that the considerations of calculus come into play both in the proof of Kepler's second law and also in the computation of $\lim\limits_{Q \to P} \frac{QR}{QT^2}$. While Newton's approach is geometric, it does not make use of analytic geometry. Hence it does not "look" like modern calculus. This is the reason why some historians of science

make a point of saying that Newton does not make use of calculus in the *Principia*.

[3] It has already been pointed out that Newton's arguments are often terse and difficult to penetrate. In reference to the demonstration just described he outdoes even himself, devoting all of two sentences to this matter. So Newton provides, at best, an outline of a demonstration. Indeed, the reconstruction of a proof based on this outline is a very delicate task. In the article: Reading the Master: Newton and the Birth of Celestial Mechanics, *Amer. Math. Monthly 104 (1997)*, 1–19, B. Pourcia takes up this task and provides the relevant (and interesting) historical perspectives as he does. In the 1960s Richard Feynman, the Nobel prize winning physicist, well aware of the very demanding nature of Newton's arguments, developed his own "elementary" approach to Newton's theory. In lecture notes that were "lost" for many years in an unoccupied office at the California Institute of Technology (where Feynman had served on the faculty), Feynman combined Newton's ideas with delicate properties of the ellipse, to deduce from the inverse square law that the planetary orbits are ellipses. For the full story and its background refer to D.L. Goodstein and J.R. Goodstein, *Feynman's Lost Lecture*, W.W. Norton and Co., New York, 1996.

[4] This explains the slight variations (of up to ten minutes, or 0.007 days) in the lengths of the seasons. Refer to the data given in Section 4.9.

[5] To get an insight into just how complicated this problem is and what an interesting history it has, refer to the book F. Diacu and P. Holmes, *Celestial Encounters, The Origins of Chaos and Stability*, Princeton University Press, Princeton, NJ, 1996.

[6] The famous Cavendish laboratories in Cambridge, England, would later be named in his honor.

[7] Laplace's celebrated five-volume treatise *Mécanique Céleste*, which appeared between 1799 and 1825, included these results and much more. Its purpose was to provide a comprehensive treatment of all that had been done in gravitational astronomy since the time of Newton. This work earned Laplace the title of "Newton of France," but it owes a considerable debt to the profound mathematical discoveries of Lagrange, who was the superior mathematician.

[8] The classic account, Herbert Butterfield, *The Origins of Modern Science*, Free Press, New York, 1965, provides the details behind these developments.